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Riemannian Geometry

sheet 02

Exercise 1. Show that on a compact manifold, every Riemannian metric is geodesically complete.

Exercise 2. Give an example of a Riemannian manifold (M, g) with associated distance function d_M and a submanifold $N \subseteq M$ with induced metric and associated distance function d_N s.t. for any two distinct points $p \neq q \in N$ there is an inequality $d_M(p, q) < d_N(p, q)$.

Exercise 3. Consider the upper half plane $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with Riemannian metric defined via $g^{hyp}_{(x,y)} := \frac{1}{y^2} \cdot g^{Euc}_{(x,y)}$, where g^{Euc} is the restriction of the standard Euclidean metric on $\mathbb{R}^2$ to $\mathbb{H}$. Determine all geodesics in $(\mathbb{H}, g^{hyp})$ and deduce that $(\mathbb{H}, g^{hyp})$ is complete.

Exercise 4. (Surfaces of revolution, continued) We use the same notations as in exercise 4 from the prvious sheet.

1. Show that all meridians, i.e. curves parametrized by arc length of the form $t \mapsto \varphi(C, v(t))$ for some constant C, are geodesics.
2. Show that all parallels, i.e. curves parametrized by arc length of the form $t \mapsto \varphi(u(t), C)$ for some constant C, are geodesics if and only if $\dot{f}(C) = 0$.
3. (Clairaut's relation) Assume $\gamma(t) = \varphi(u(t), v(t))$ is a curve parametrized by arc length which is never tangent to a parallel, i.e. $\dot{v}(t) \neq 0$ for all t. Let $r(t) = f(v(t))$ denote the distance of the axis of rotation to the parallel passing through $\gamma(t)$. Let $\psi(t)$ denote the angle between α and this parallel, i.e. for $\varphi_1 := \frac{\partial \varphi}{\partial u}$, one has $\|\varphi_1\| \cdot \|\dot{\gamma}\| \cdot \cos\psi = \langle \dot{\gamma}, \varphi_1 \rangle$. Show that γ defines a geodesic if and only if $r(t) \cdot \cos\psi(t)$ is a constant function.

   Hint: Use $\frac{\partial log k(t)}{\partial t} = \frac{\dot{k}}{k}$.
4. Draw pictures illustrating these results.