5.5 Fredholm alternative and the spectral theorem for compact operators

General assumption: H is a separable Hilbert space over $\mathbb{C}$.

Motivation 5.32

(i) Let M be an $N \times N$-matrix (over $\mathbb{C}$) for $N \in \mathbb{N}$, and let $z \in \mathbb{C}$.

Then: Either: $M\varphi = z\varphi$ has a solution $0 \neq \varphi \in \mathbb{C}^N$

Or: $(M-z)^{-1}$ exists

(ii) Let $H = L^2([0,1])$ and $M \in BL(L^2([0,1]))$ be defined by

$$(M\varphi)(x) := x\varphi(x) \quad \text{for a.e. } x \in [0,1] \ \forall \varphi \in L^2([0,1]).$$

(a) $M\varphi = z\varphi$ has no solutions $0 \neq \varphi \in L^2$ since:

$x\varphi(x) = z\varphi(x)$ a.e. $x \in [0,1] \Rightarrow \varphi(x) = 0$ a.x $\in [0,1]$

(b) $(M-z)$ is not surjective $\forall z \in [0,1]$: (so, $(M-z)^{-1}$ does not exist).

- $((M-z)\varphi)(x) = (x-z)\varphi(x)$
- Any element $\varphi \in L^2([0,1])$ which has a representative which is (1) continuous at $x = z$ (2) with $\varphi(z) \neq 0$ satisfies $\varphi \notin \operatorname{ran}(M-z)$: Assume $\varphi \in \operatorname{ran}(M-z)$.

Then there exists $\psi \in L^2([0,1])$ s.t. $(M-z)\psi = \varphi$, hence $\psi(x) = \frac{1}{x-z}\varphi(x)$ (a.e. x) – which is not an element of $L^2([0,1])$ ↯.

(c) For compact operators, the above either/or-statement in (a) still holds (see Cor. 5.34).

Theorem 5.33 (Analytic Fredholm theorem)

Let $D \subseteq \mathbb{C}$ be open and connected, let $f: D \to BL(H)$ be (norm) analytic, and $f(z)$ compact $\forall z \in D$ $(f(z) \in K(H))$.

Then: Either (A1): $(f(z) - \mathbb{1})^{-1}$ does not exist for any $z \in D$

or (A2): There exists a discrete subset $S \subseteq D$ without accumulation points in D, such that the map

$$D \setminus S \to BL(H) \qquad \text{is well-defined and analytic.}$$
$$z \mapsto (f(z) - \mathbb{1})^{-1}$$

Moreover: $z \in S$ iff $f(z)\varphi = \varphi$ has a solution $0 \neq \varphi \in H$.

Corollary 5.34 Let $T \in BL(H)$ be compact (i.e. $T \in K(H)$) and $z \in \mathbb{C} \setminus \{0\}$. Then

Either: $T\varphi = z\varphi$ has a solution $0 \neq \varphi \in H$

or: $(T - z\mathbb{1})^{-1} \in BL(H)$ exists.

Pf (5.34): Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z} T = \frac{T}{z}$.
Alternative (A1) does not hold in Thm. 5.33, since $\frac{T}{z} - \mathbb{1}$ has an inverse in $BL(H)$ $\forall |z| > \|T\|$ (Neumann series!)
Hence, Alternative (A2) holds. $\square$

Pf (5.33): It suffices to prove: $\forall z_0 \in D$ $\exists$ neighbourhood $\mathcal{N}(z_0)$ s.t. either (A1) or (A2) holds on $\mathcal{N}(z_0)$. [Indeed: $\forall z_0 \in D$, colour $\mathcal{N}(z_0)$ red or blue, depending on which alternative holds; D is connected, i.e. if both colours appear, then there exists $\tilde{z}_0$ s.t. $\mathcal{N}(\tilde{z}_0)$ cannot be uniquely coloured – contradiction]

Fix $z_0 \in D$. As f continuous: $\exists r > 0$: $\|f(z) - f(z_0)\| < \frac{1}{2}$ for all $z \in D_r := B_r(z_0)$. By Thm. 5-31 (separability!), $\exists F \in BL(H)$, finite rank, s.t. $\|f(z_0) - F\| < \frac{1}{2}$. Hence $\|f(z) - F\| < 1$, and so (by 5-18 (Neumann series))

$$(1) \qquad \begin{aligned} D_r &\to BL(H) \\ z &\mapsto (\mathbb{1} - f(z) + F)^{-1} \end{aligned} \qquad \text{is well-defined and analytic.}$$

Let $\quad g(z) := F(\mathbb{1} - f(z) + F)^{-1}$, let $N := \operatorname{rank}(F) < \infty$, and let $\varphi_1, \dots, \varphi_N \in H$ be an ONB of $\operatorname{ran}(F)$.

Then $\operatorname{ran}(g(z)) \subseteq \operatorname{ran}(F)$, so: $\forall \varphi \in H$ (expand $g(z)\varphi$ in ONB)

$$(2) \qquad g(z)\varphi = \sum_{n=1}^{N} \langle \varphi_n, g(z)\varphi \rangle \varphi_n$$

Note also that

$$(3) \qquad f(z) - \mathbb{1} = (g(z) - \mathbb{1})(\mathbb{1} - f(z) + F)$$

For all $z \in D_r$ we have the following equivalences:

$$f(z) - \mathbb{1} \text{ \underline{not} invertible} \overset{(1),(3)}{\iff} g(z) - \mathbb{1} \text{ \underline{not} invertible}$$

$$\Updownarrow (*) \quad (\text{Pf. below!})$$

$$\exists \text{ sol. } 0 \neq \varphi \in H \text{ of } f(z)\varphi = \varphi \iff \exists \text{ sol. } 0 \neq \varphi \in H \text{ of } g(z)\varphi = \varphi$$

$$\Updownarrow (2) \quad \varphi = \sum_{n=1}^{N} \beta_n \varphi_2$$

$$\exists \beta_1, \dots, \beta_N \in \mathbb{C} \text{ (not \underline{all} } = 0): \beta_n = \sum_{m=1}^{N} \underbrace{\langle \varphi_n, g(z)\varphi_m \rangle}_{=: A_{nm}(z)} \beta_m$$

$$\Updownarrow$$

$$d(z) := \det(A(z) - \mathbb{1}_{N \times N}) = 0, \text{ where } A := (A_{nm})_{1 \le n,m \le N}$$

<u>But</u>: $d: D_r \to \mathbb{C}$ is <u>analytic</u>, because, by (1), A_{nm} is analytic $\forall n,m$

By the identity thm (Complex Analysis):

<u>Either</u> $S_r := \{z \in D_r \mid d(z) = 0\}$ has no accum. point in D_r

i.e. (A2) holds on D_r

or $S_r = D_r$, i.e. (A1) holds on D_r; $\Rightarrow$ claim!

It remains to prove (*): The pb. is reduced to the finite rank case.

"$\Uparrow$": Clear.

"$\Downarrow$": By contradiction. Assume $d(z) \neq 0$. Then:

$\forall \xi \in \mathbb{C}^N : (A(z) - \mathbb{1}_{N\times N})\beta = \xi$ has a solution $\beta \in \mathbb{C}^N$.

For an arbitrary (fixed) $\varphi \in \mathcal{H}$, choose $\xi_n = \xi_n^\varphi := \langle \varphi_n, g(z)\varphi\rangle$ for $n = 1, \dots, N$, and let $\beta^\varphi \in \mathbb{C}^N$ be a corresponding solution.

Then: $\tilde{\varphi} := -\varphi + \sum_{n=1}^N \beta_n^\varphi \varphi_n$ solves $(g(z) - \mathbb{1})\tilde{\varphi} = \varphi$,

since:

$$(g(z) - \mathbb{1})\tilde{\varphi} = \varphi - g(z)\varphi + (g(z) - \mathbb{1})\sum_{m=1}^N \beta_m^\varphi \varphi_m$$

$$\overset{(2)}{=} \varphi + \sum_{n=1}^N \underbrace{\Big[-\underbrace{\langle \varphi_n, g(z)\varphi\rangle}_{\xi_n^\varphi} + \sum_{m=1}^N (A_{nm}(z) - \delta_{nm})\beta_m^\varphi \Big]}_{=0} \varphi_n$$

$\Rightarrow g(z) - \mathbb{1}$ is invertible $\lightning$ ∎

Corollary 5.35 (Riesz-Schauder) Let $T \in BL(H)$ be compact. Then $\sigma(T)$ has no accumulation point in $\mathbb{C}$, except possibly at 0. Moreover, any $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity (i.e., the corresponding eigenspace is finite-dim.)

Pf: Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z}$ in Thm. 5.33.

Then $\mathbb{C} \setminus (\{0\} \cup S) \subseteq \rho(T)$ so $\sigma(T) \subseteq \{0\} \cup S$; also, $S \subseteq \sigma_p(T) \subseteq \sigma(T) \subseteq \{0\} \cup S$. Remove 0 from the sets in this chain of inclusions and note that $S = S \setminus \{0\}$.

Hence $S \subseteq \sigma_p(T) \setminus \{0\} \subseteq \sigma(T) \setminus \{0\} \subseteq S$, and so

$$S = \sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

Also, 0 is the only possible accumulation point of S by 5.33.

<u>Finite multiplicity</u>: Suppose <u>not</u> (only possible for $\dim(H) = \infty$). Then there exists an infinite ONB $\{\varphi_n\}_{n\in\mathbb{N}}$ of the eigenspace corresponding to some eigenvalue $\lambda \neq 0$, and $T\varphi_n = \lambda\varphi_n$ for all n (eigenspace <u>closed</u>!). Compactness of T implies that $(T\varphi_n)_n = (\lambda\varphi_n)_n$ has a <u>convergent</u> subsequence, which is a contradiction, since $\lambda \neq 0$ and $\{\varphi_n\}_n$ ONB. ∎

Remark 5.36 (a) If $\dim H = \infty$, then $0 \in \sigma(T)$

<u>Pf</u>: If <u>not</u>, then $0 \in \rho(T)$, i.e. $T^{-1} \in BL(H)$ and $\mathbb{1} = TT^{-1}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an ONB of H. Then $\varphi_n = T(T^{-1}\varphi_n)$ $\forall n$. Since $\{T^{-1}\varphi_n\}_n \subseteq H$ is <u>bounded</u>, and T compact, $\{\varphi_n\}_n$ has a convergent subsequence ↯ (since $\{\varphi_n\}_n$ ONB). ∎

(b) Corollaries 5.34 & 5.35 have generalisations to <u>Banach spaces</u>.

In the following we use the convention $n = 1, \dots, \infty$: $\Leftrightarrow n \in \mathbb{N}$ (to have <u>unified</u> notation).

Theorem 5.37 (Singular-value decomposition for compact operators)

Let $T \in BL(H)$ be compact. Then there exists $N \in \mathbb{N}_0 \cup \{\infty\}$, orthonormal sets $\{\varphi_n\}_{n=1,\dots,N}$, $\{\psi_n\}_{n=1,\dots,N} \subseteq H$ (which are <u>not</u> necessarily ONB's) and <u>singular values</u> $(\lambda_n)_{n=1,\dots,N} \subseteq (0,\infty)$ s.t.

$$T = \sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot \rangle \psi_n . \qquad (*)$$

If $N = \infty$, then $(\lambda_n)_{n\in\mathbb{N}}$ is a null sequence, and the sum in $(*)$ converges in the operator norm.

The proof relies on

Theorem 5.38 (Hilbert-Schmidt theorem; Spectral theorem for self-adjoint and compact op's on Hilbert space)

Let $T \in BL(H)$ be self-adjoint and compact. Then there exists an ONB $\{\varphi_n\}_{n=1,\dots,\dim H} \subseteq H$ of eigenvectors, and eigenvalues $(\lambda_n)_{n=1,\dots,\dim H} \subseteq \mathbb{R}$ (listed according to their multiplicities) such that

$$T\varphi_n = \lambda_n \varphi_n \qquad \forall n = 1, \dots, \dim H.$$

Moreover,

$$T = \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n. \qquad (*)$$

If $\dim H = \infty$, then $(\lambda_n)_{n\in\mathbb{N}}$ is a nullsequence, and the sum in $(*)$ converges in the operator norm.

Pf of 5.38: (i) Let $0 \neq \mu_j$, $j = 1, 2, \dots$ be the pairwise distinct non-zero (real!) eigenvalues of T, i.e., $\mu_j \neq \mu_k$ for $j \neq k$. For every j, choose an ONB B_j in the finite-dim. (Cor 5.35) eigenspace of T corresponding to μ_j. Then $B := \bigcup_j B_j$ is an orthonormal set in H, since eigenvectors corresp. to different eigenvalues of normal operators are orthogonal (check!).

Let $H_0 := \overline{\operatorname{span}(B)}$. Then $T(H_0) \subseteq H_0$, and as $T = T^*$ this gives $T(H_0^\perp) \subseteq H_0^\perp$:

$$\forall x_\perp \in H_0^\perp \ \forall x_0 \in H_0 : \langle x_0, T x_\perp \rangle \overset{T=T^*}{=} \langle \underbrace{T x_0}_{\in H_0}, x_\perp \rangle = 0 \quad \Rightarrow \quad T x_\perp \perp H_0$$

Let $T^\perp := T|_{H_0^\perp} : H_0^\perp \to H_0^\perp$. Then $T^\perp$ is selfadjoint & compact

But $\sigma(T^\perp) = \{0\} \Rightarrow$ spectral radius $r(T^\perp) = 0$.

So, by 5.25(b): $\|T^\perp\| = 0$, hence $T^\perp = 0$, and so $H_0^\perp \subseteq \ker(T)$ (in fact "=").

So: if $B^\perp$ is any ONB of $H_0^\perp$, then $B \cup B^\perp =: \{\varphi_n\}_{n=1,\dots,\dim H}$ is ONB of H consisting of eigenvectors of T.

(ii) Let λ_n be the eigenvalue corresponding to φ_n, $n = 1, \dots, \dim H$. Then $(\lambda_n)_{n=1,\dots,\dim H}$ is sequence of eigenvalues enumerated according to multiplicities and (if $\dim H = \infty$), a null sequence by Cor. 5.35

(iii) $T \overset{(*)}{=} \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$ holds because, if $\dim H < \infty$, both sides obviously agree on ONB $\{\varphi_n\}_{n=1,\dots,\dim H}$:

$T\varphi_m = \lambda_m \varphi_m \ \forall m \in \mathbb{N}$ and (finite sum!)

$$\sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \varphi_m \rangle \varphi_n = \lambda_m \varphi_m \quad \text{(ONB!)}$$

If $\dim H = \infty$, we first prove that the RHS in $(*)$ is a Cauchy seq. wrt. the operator norm. Indeed, let $\varepsilon > 0$. Since $(\lambda_n)_{n\in\mathbb{N}}$ is a null sequence, $\exists M_1 \in \mathbb{N}$ s.t. $\sup_{n \geq M_1} |\lambda_n| < \varepsilon$. Hence, $\forall M_2 \geq M_1$:

$$\sup_{\|\varphi\|=1} \left\| \sum_{n=M_1+1}^{M_2} \lambda_n \langle \varphi_n, \varphi \rangle \varphi_n \right\|^2 = \sup_{\|\varphi\|=1} \left[\sum_{n=M_1+1}^{M_2} |\lambda_n| |\langle \varphi_n, \varphi \rangle| \right]$$

$$\leq \sup_{\|\varphi\|=1} \Big[\underbrace{\Big(\sup_{n\geq M_1} |\lambda_n|^2\Big)}_{<\varepsilon^2} \underbrace{\sum_{n=M_1+1}^{M_2} |\langle \varphi_n, \varphi\rangle|^2}_{\leq \|\varphi\|^2 \text{ (Bessel)}} \Big] < \varepsilon^2$$

Since $BL(H)$ is complete, $\exists R \in BL(H)$ s.t. $R = \sum_{n\in\mathbb{N}} \lambda_n \langle \varphi_n, \cdot\rangle \varphi_n$. Since both T & R are bounded, it suffices to prove that they agree on a dense subset of H (Bounded Linear Extension Thm. 2.31). But as R is a uniform (thus strong) limit, we have

$$R\varphi_m = \Big[\lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot\rangle \varphi_n\Big]\varphi_m$$

$$= \lim_{N\to\infty} \Big[\underbrace{\Big(\sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot\rangle \varphi_n\Big)\varphi_m}_{\sum_{n=1}^{N} \lambda_n \langle \varphi_n, \varphi_m\rangle \varphi_n} \Big] = \lambda_m \varphi_m \quad \forall m \in \mathbb{N}$$

Hence, T and R agree on ONB, so they agree on $\operatorname{span}\{\varphi_m \mid m\in\mathbb{N}\} \subseteq H$, which is dense in H ∎

<u>Pf of 5.37</u>: Since T is compact, so is T^*T (by 5.30(c)). Since T^*T is also self-adjoint (check!), by 5.38: $\exists$ orthonormal set $\{\varphi_n\}_{n=1,\dots,\dim H} \subseteq H$ and $(\tilde\lambda_n)_{n=1,\dots,\dim H} \subseteq \mathbb{R}$ such that

$$T^*T = \sum_{n=1}^{\dim H} \tilde\lambda_n \langle \varphi_n, \cdot\rangle \varphi_n.$$

Now, since $0 \leq \|T\varphi\|^2 = \langle \varphi, T^*T\varphi\rangle = \sum_{n=1}^{\dim H} \tilde\lambda_n |\langle \varphi_n, \varphi\rangle|^2 \quad \forall \varphi \in H$ we have (take $\varphi = \varphi_m$) that $\tilde\lambda_m \geq 0 \quad \forall m \in \mathbb{N}$. Terms with $\tilde\lambda_n = 0$ do not contribute to (*) in 5.38. Re-name the eigenvalues $(\tilde\lambda_n)_{n=1,\dots,\dim H}$ such that $(\tilde\lambda_n)_{n=1,\dots,N} \subseteq (0,\infty)$ be the collection of non-zero eigenvalues,

and $(\tilde{\lambda}'_{n'} = 0)_{n'=1,\dots,N'}$ are the zero eigenvalues, where $N, N' \in \mathbb{N}_0 \cup \{\infty\}$. Let $\{\psi_n \mid n = 1,\dots,N\}$ and $\{\psi'_{n'} \mid n' = 1,\dots,N'\}$ be the set of the corresponding eigenvectors, which, together, form the ONB of H.

Hence

$$T^*T = \sum_{n=1}^{N} \tilde{\lambda}_n \langle \psi_n, \cdot \rangle \psi_n \quad \text{and} \quad \ker(T) = \overline{\operatorname{span}\{\psi'_{n'} \mid n = 1,\dots,N'\}}.$$

Set $\lambda_n := \tilde{\lambda}_n^{1/2} = \sqrt{\tilde{\lambda}_n}$ and $\varphi_n = \frac{1}{\lambda_n} T\psi_n$ for all $n = 1,\dots,N$

Then $T\psi_n = \lambda_n \varphi_n$, and

$$\langle \varphi_m, \varphi_n \rangle = (\lambda_m \lambda_n)^{-1} \langle \psi_m, \underbrace{T^*T\psi_n}_{\tilde{\lambda}_n \psi_n} \rangle = \delta_{mn} \quad \forall m,n = 1,\dots,N$$

That is, $\{\varphi_n\}_{n=1,\dots,N} \subseteq H$ is an orthonormal set. Note that if $N = \infty$ (only possible if $\dim(H) = \infty$) then $(\lambda_n)_{n \in \mathbb{N}}$ is a null-sequence by 5.38.

Finally, the same (!) argument as in the proof of 5.38 gives

$$T = \sum_{n=1}^{N} \lambda_n \langle \psi_n, \cdot \rangle \varphi_n$$

with the sum being uniformly (i.e. norm) convergent if $N = \infty$ (use the ONB $\{\psi_1,\dots,\psi_N\} \cup \{\psi'_1,\dots,\psi'_{N'}\}$ to test the equality) ∎