Prove of the existence of a solution to a simplified differential equation for interacting bose gases.

Markus Wiener

An effective theory for interacting Bose gases

26.05.2021

Introduction

Theorem

In this talk we are going to prove Theorem 1.3 of the paper „Analysis of a simple equation for the ground state energy of the Bose gas“ by Carlen, Jauslin, Lieb. It states:

Theorem

In this talk we are going to prove Theorem 1.3 of the paper „Analysis of a simple equation for the ground state energy of the Bose gas“ by Carlen, Jauslin, Lieb. It states:

Theorem

Let $d \in \mathbb{N}, p > \max\left\{\frac{d}{2}, 1\right\}$ and $\mathcal{V} \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ non-negative. Then there is a continuous function $\rho(e)$ on $(0, \infty)$ satisfying

- $\lim_{e \to 0} \rho(e) = 0$,
- $\lim_{e \to \infty} \rho(e) = \infty$,
- *there exists a unique integrable function $u(x)$ on $\mathbb{R}^d$ with $0 \leq u(x) \leq 1$ for all $x \in \mathbb{R}^d$, which solves the system of equations*

$$\begin{aligned} \big(-\triangle + 4e + \mathcal{V}(x)\big)u(x) &= \mathcal{V}(x) + 2e\rho(e)(u * u)(x) \\ e &= \frac{\rho(e)}{2} \int (1 - u(x))\mathcal{V}(x)\, dx. \end{aligned} \tag{1}$$

Overwiev

1. Rewrite equation (1) in a better suited form,
2. Define for fixed e sequences (ρ_n) and (u_n),
3. Prove some properties of (ρ_n) and (u_n),
4. Prove that the limits ρ and u of these sequences exist and solve the system of equations (1),
5. Prove that $\rho(e)$ is continuous and has the desired limit-properties.

Overwiev

1. Rewrite equation (1) in a better suited form,
2. Define for fixed e sequences (ρ_n) and (u_n),
3. Prove some properties of (ρ_n) and (u_n),
4. Prove that the limits ρ and u of these sequences exist and solve the system of equations (1),
5. Prove that $\rho(e)$ is continuous and has the desired limit-properties.
6. (Prove uniqueness of $\rho(e)$ and u)

Rewriting equation (1)

Concept: Green's function

Let L be a linear differential operator. Then its Green function $G(x, s)$ is defined via

$$LG(x, s) = \delta(x - s).$$

Concept: Green's function

Let L be a linear differential operator. Then its Green function $G(x,s)$ is defined via

$$LG(x,s) = \delta(x-s).$$

Claim:

$$Lu(x) = f(x) \Rightarrow \int G(x,s)f(s)\text{ds is a solution to the DE.}$$

Prove:

$$L\int G(x,s)f(x)\ \text{ds} = \int LG(x,s)f(x)\ \text{ds} = \int \delta(x-s)f(s)\ \text{ds} = f(x).$$

Concept: Green's function

Let L be a linear differential operator. Then its Green function $G(x, s)$ is defined via

$$LG(x, s) = \delta(x - s).$$

Claim:

$$Lu(x) = f(x) \Rightarrow \int G(x, s) f(s) \mathrm{ds} \text{ is a solution to the DE.}$$

Prove:

$$L \int G(x, s) f(x) \text{ ds} = \int LG(x, s) f(x) \text{ ds} = \int \delta(x - s) f(s) \text{ ds} = f(x).$$

Notation:

$$L^{-1} = G := \int G(x, s)(\cdot)\mathrm{ds}$$

Concept: Contraction semigroup

A strongly continuous semigroup on a Banach space X is a family of bounded, linear operators $(T_i)_{i\in\mathbb{R}_+}$ on X, such that

- $T(0) = Id_X$,
- $\forall t, s \geq 0 : T_{t+s} = T_t T_s$,
- $\forall x \in X : ||T_t x - x|| \to 0, \text{ as } t \to 0.$

A strongly continuous semigroup is called a contraction semigroup, if for all $t \in \mathbb{R}_+$, one has $||T_t|| \leq 1$

Concept: Contraction semigroup

A strongly continuous semigroup on a Banach space X is a family of bounded, linear operators $(T_i)_{i\in\mathbb{R}_+}$ on X, such that

- $T(0) = Id_X$,
- $\forall t, s \geq 0 : T_{t+s} = T_t T_s$,
- $\forall x \in X : ||T_t x - x|| \to 0, \text{ as } t \to 0.$

A strongly continuous semigroup is called a contraction semigroup, if for all $t \in \mathbb{R}_+$, one has $||T_t|| \leq 1$

A generator G of a strongly continuous semigroup (T_t) is defined by

$$Gx := \lim_{t\to 0} -\frac{1}{t}(T_t - Id_X)x.$$

This operator must not be exist for all $x \in X$. The set of all $x \in X$, such that G exists is called the domain $D(G)$ of G.

Ressources

Ressources for the claims:

E.H. Lieb, M. Loss. *Analysis*. Second edition, Graduate studies in mathematics, Americal Mathematical Society (2001).

M. Reed, B. Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness,*. second edition, Academic Press, New York (1975).

Assumed results

Theorem:

$-\triangle + 4e$ for $e > 0$ is a bijection between $W^{2,p}$ and L^p with inverse $G := (-\triangle + 4e)^{-1}$ given by

$$Gu = Y_{4e} * u,$$

where Y_{4e} is the Yukawa potential. The Yukawa potential is non-negative and

$$\int Y_{4e}(x)\mathrm{dx} = \frac{1}{4e}.$$

Intermezzo: Sobolev-spaces

$W^{2,p}(\mathbb{R}^d)$ is the Sobolev-space of order 2 over $L^p(\mathbb{R}^d)$, that is the set of all $f \in L^p(\mathbb{R}^d)$, such that for all $\alpha \in \{1, ..., d\}$ and $(\beta_1, \beta_2) \in \{1, ..., d\} \times \{1, ..., d\}$

$$\frac{\partial f}{\partial \alpha} \in L^p(\mathbb{R}^d) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial \beta_1 \partial \beta_2} \in L^p(\mathbb{R}^d).$$

It is equipped with the Sobolev-norm, defined as

$$\begin{aligned} ||f||_{W^{2,p}} &= \sum_{|\alpha| \leq 2} ||D^\alpha f||_p \\ &= ||f||_p + \sum_{\alpha \in \{1,\dots,d\}} \left|\left| \frac{\partial}{\partial x^\alpha} f \right|\right|_p + \sum_{\alpha \in \{1,\dots,d\}} \left|\left| \frac{\partial^2}{\partial x^\alpha \partial x^\alpha} f \right|\right|_p + \sum_{\substack{\alpha,\beta \in \{1,..,d\} \\ \alpha \neq \beta}} \left|\left| \frac{\partial^2}{\partial x^\alpha \partial x^\beta} f \right|\right|_p . \end{aligned}$$

Assumed results

Theorem:

$-\triangle + 4e$ for $e > 0$ is a bijection between $W^{2,p}$ and L^p with inverse $G := (-\triangle + 4e)^{-1}$ given by

$$Gu = Y_{4e} * u,$$

where Y_{4e} is the Yukawa potential. The Yukawa potential is non-negative and

$$\int Y_{4e}(x)\mathrm{dx} = \frac{1}{4e}.$$

Theorem:

$(-\triangle + 4e)$ is the generator of a contraction semigroup with domain $D(-\triangle + 4e) = W^{2,p}(\mathbb{R}^d)$. The contraction semigroup is positivity preserving, that is

$$u \geq 0 \Rightarrow e^{(\triangle - 4e)t} u \geq 0$$

for all $t \geq 0$.

Rewriting equation (1)

Alternative form 1

We start with:

$$big(-\triangle +4e+\mathcal{V}(x)\big)u(x)=\mathcal{V}(x)+2e\rho(e)(u*u)(x) \tag{2}$$

Rewriting equation (1)

Alternative form 1

We start with:

$$big(-\triangle + 4e + \mathcal{V}(x)\big)u(x) = \mathcal{V}(x) + 2e\rho(e)(u*u)(x) \tag{2}$$

Rearranging this gives

$$\big(-\triangle + 4e\big)u(x) = \mathcal{V}(x)\big(1 - u(x)\big) + 2e\rho(e)(u*u)(x).$$

Rewriting equation (1)

Alternative form 1

We start with:

$$big(-\triangle+4e+\mathcal{V}(x)\big)u(x)=\mathcal{V}(x)+2e\rho(e)(u*u)(x) \tag{2}$$

Rearranging this gives

$$\big(-\triangle+4e\big)u(x)=\mathcal{V}(x)\big(1-u(x)\big)+2e\rho(e)(u*u)(x).$$

Acting with $G_e=(-\triangle+4e)^{-1}$ gives

$$u(x)=Y_{4e}*\big(\mathcal{V}(1-u)\big)(x)+2e\rho(e)(Y_{4e}*u*u)(x).$$

Alternative form 1

We start with:

$$big(-\triangle+4e+\mathcal{V}(x)\big)u(x)=\mathcal{V}(x)+2e\rho(e)(u*u)(x) \tag{2}$$

Rearranging this gives

$$\big(-\triangle+4e\big)u(x)=\mathcal{V}(x)\big(1-u(x)\big)+2e\rho(e)(u*u)(x).$$

Acting with $G_e=(-\triangle+4e)^{-1}$ gives

$$u(x)=Y_{4e}*\big(\mathcal{V}(1-u)\big)(x)+2e\rho(e)\big(Y_{4e}*u*u\big)(x).$$

Alternative form 2

Alternatively, acting with $K_e=(-\triangle+4e+\mathcal{V})^{-1}$ on (2) immediately gives

$$u(x)=K_e\mathcal{V}(x)+2e\rho_eK_e(u*u)(x).$$

Theorem

For the following theorem, see *Reed, Simon, page 244*.

Theorem (1)

Let A be the generator of a contraction semigroup on a Banach space X. Suppose that B is an accretive operator, with $D(A) \subset D(B)$ and

$$||B\phi|| \leq a||A\phi|| + b||\phi||$$

for some $b \in \mathbb{R}_+$ and some $a < \frac{1}{2}$ and all $\phi \in D(a)$. Then $A + B$ (defined on $D(A + B) = D(A)$) is a closed accretive operator, which generates a contraction semigroup.

Application of Theorem 1

- $(-\triangle + 4e)$ is the generator of a contraction semi-group, as established before.

Application of Theorem 1

- $(-\triangle + 4e)$ is the generator of a contraction semi-group, as established before.
- To prove $\mathcal{V}(x)$ to be accretive, we use the following theorem found in (Reed,Simon, page 241):

Theorem (2)

An operator A is the generator of a contraction semigroup, if and only if it is accretive and $A + \lambda Id$ is surjective for all $\lambda > 0$.

Application of Theorem 1

We define $e^{-\mathcal{V}t} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ as

$$\big(e^{-\mathcal{V}t}u\big)(x) := e^{-\mathcal{V}(x)t}u(x) = \left(\sum_{n\in\mathbb{N}_0} \frac{\big(-\mathcal{V}(x)t\big)^n}{n!}\right) u(x)$$

Application of Theorem 1

We define $e^{-\mathcal{V}t} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ as

$$(e^{-\mathcal{V}t}u)(x) := e^{-\mathcal{V}(x)t}u(x) = \left(\sum_{n\in\mathbb{N}_0} \frac{\big(-\mathcal{V}(x)t\big)^n}{n!}\right) u(x)$$

- $||T_t v| \leq ||v||$, since
  $\mathcal{V}(x) \geq 0 \Rightarrow 0 \leq e^{-\mathcal{V}(x)t} \leq 1 \Rightarrow e^{-\mathcal{V}(x)t}v(x) \leq v(x) \Rightarrow ||e^{-\mathcal{V}t}v|| \leq ||v||$

Application of Theorem 1

We define $e^{-\mathcal{V}t} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ as

$$\left(e^{-\mathcal{V}t}u\right)(x) := e^{-\mathcal{V}(x)t}u(x) = \left(\sum_{n\in\mathbb{N}_0} \frac{\left(-\mathcal{V}(x)t\right)^n}{n!}\right) u(x)$$

- $||T_t v| \leq ||v||$, since
  $\mathcal{V}(x) \geq 0 \Rightarrow 0 \leq e^{-\mathcal{V}(x)t} \leq 1 \Rightarrow e^{-\mathcal{V}(x)t}v(x) \leq v(x) \Rightarrow ||e^{-\mathcal{V}t}v|| \leq ||v||$
- $T_0 = Id_X$, since

$$\left(e^{-\mathcal{V}\cdot 0}v\right)(x) = e^{-\mathcal{V}(x)\cdot 0}v(x) = 1 \cdot v(x),$$

Application of Theorem 1

We define $e^{-\mathcal{V}t} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ as

$$\big(e^{-\mathcal{V}t}u\big)(x) := e^{-\mathcal{V}(x)t}u(x) = \left(\sum_{n\in\mathbb{N}_0} \frac{\big(-\mathcal{V}(x)t\big)^n}{n!}\right) u(x)$$

- $||T_t v| \leq ||v||$, since
  $\mathcal{V}(x) \geq 0 \Rightarrow 0 \leq e^{-\mathcal{V}(x)t} \leq 1 \Rightarrow e^{-\mathcal{V}(x)t}v(x) \leq v(x) \Rightarrow ||e^{-\mathcal{V}t}v|| \leq ||v||$

- $T_0 = Id_X$, since
  $$\big(e^{-\mathcal{V}\cdot 0}v\big)(x) = e^{-\mathcal{V}(x)\cdot 0}v(x) = 1 \cdot v(x),$$

- $T_t T_s = T_{t+s}$, since
  $$\begin{aligned}\big(e^{-\mathcal{V}\cdot t}e^{-\mathcal{V}\cdot s}v\big)(x) &= e^{-\mathcal{V}(x)\cdot t}e^{-\mathcal{V}\cdot s}v(x) \\ &= e^{-\mathcal{V}(x)\cdot(t+s)}v(x) = \big(e^{-\mathcal{V}(t+s)}v\big)(x)\end{aligned}$$

Application of Theorem 1

- $\lim_{t\to 0}||T_t v - v|| = 0$, since $e^{-\mathcal{V}(x)t}v(x)$ converges for $t \to 0$ pointwise towards $v(x)$ and since $e^{-\mathcal{V}(x)t}v(x) \leq v(x)$, by dominated convergence we have

$$\lim_{t\to 0}||T_t v - v|| = ||\lim_{t\to 0} T_t v - v|| = 0$$

Application of Theorem 1

- $\lim_{t\to 0} ||T_t v - v|| = 0$, since $e^{-\mathcal{V}(x)t}v(x)$ converges for $t \to 0$ pointwise towards $v(x)$ and since $e^{-\mathcal{V}(x)t}v(x) \leq v(x)$, by dominated convergence we have

$$\lim_{t\to 0} ||T_t v - v|| = || \lim_{t\to 0} T_t v - v|| = 0$$

- $\mathcal{V} = \lim_{t\to 0} -\frac{1}{t}\left(T_t - Id_X\right)v$, since for all $v \in L^p(\mathbb{R}^d)$, such that $\mathcal{V} \cdot v \in L^p(\mathbb{R}^d)$:

$$\begin{aligned}\lim_{t\to 0} -\frac{1}{t}\left(e^{-\mathcal{V}(x)t} - 1\right)v(x) &= \left(\lim_{t\to 0} -\frac{1}{t}\left(e^{-\mathcal{V}(x)t} - 1\right)\right)v(x) \\ &= \mathcal{V}(x)v(x).\end{aligned}$$

Application of Theorem 1

- $\lim_{t\to 0} ||T_t v - v|| = 0$, since $e^{-\mathcal{V}(x)t}v(x)$ converges for $t \to 0$ pointwise towards $v(x)$ and since $e^{-\mathcal{V}(x)t}v(x) \leq v(x)$, by dominated convergence we have

$$\lim_{t\to 0} ||T_t v - v|| = || \lim_{t\to 0} T_t v - v|| = 0$$

- $\mathcal{V} = \lim_{t\to 0} -\frac{1}{t}\left(T_t - Id_X\right)v$, since for all $v \in L^p(\mathbb{R}^d)$, such that $\mathcal{V} \cdot v \in L^p(\mathbb{R}^d)$:

$$\begin{aligned}\lim_{t\to 0} -\frac{1}{t}\left(e^{-\mathcal{V}(x)t} - 1\right)v(x) &= \left(\lim_{t\to 0} -\frac{1}{t}\left(e^{-\mathcal{V}(x)t} - 1\right)\right)v(x)\\ &= \mathcal{V}(x)v(x).\end{aligned}$$

Conclusion: $\mathcal{V}(x)$ is the generator of a contraction semigroup and therefore accretive.

Application of Theorem 1

- $D(-\triangle + 4e) \subseteq D(\mathcal{V})$

  $D(\mathcal{V}) = \{u \in L^p(\mathbb{R}^d) | V \cdot u \in L^p(\mathbb{R}^d)\}$, especially all bounded u.
  Since all $u \in W^{2,p}(\mathbb{R}^d)$ are bounded

$$D(-\triangle + 4e) = W^{2,p}(\mathbb{R}^d) \subset D(-\mathcal{V}).$$

Application of Theorem 1

- $D(-\triangle +4e) \subseteq D(\mathcal{V})$

$D(\mathcal{V}) = \{u \in L^p(\mathbb{R}^d) | V \cdot u \in L^p(\mathbb{R}^d)\}$, especially all bounded u. Since all $u \in W^{2,p}(\mathbb{R}^d)$ are bounded

$$D(-\triangle +4e) = W^{2,p}(\mathbb{R}^d) \subset D(-\mathcal{V}).$$

Note: Boundedness follows from the Sobolev embedding

$$W^{k,p}(\Omega) \subseteq C(\Omega)$$

for bounded Lipschitz-domains Ω and $kp > n$ (which is satisfied by assumption of p in the theorem) , together with the fact, that functions in $W^{2,p}(\mathbb{R}^d)$ go to 0 at infinity. Boundedness also holds for $W^{2,1}(\mathbb{R}^d)$ as for $kp < n$

$$W^{k,p}(\Omega) \subset L^q(\Omega) \quad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{k}{p},$$

where we can use $W^{2,1}(\Omega) \subset W^{1,1}(\Omega)$, to make $q = \infty$ (for this, we have to assume $d > 1$), see [Sobolev Spaces and Elliptic Equations, Long Chen, page 8].

Application of Theorem 1

Last to check: Bound

For this let $\varepsilon > 0$ and $M \subseteq \mathbb{R}^d$, such that

$$\mathbb{1}_M \mathcal{V}(x) \leq C \quad \text{and} \quad ||\mathbb{1}_{M^c} \mathcal{V}||_p \leq \varepsilon.$$

With this, we can calculate for every $f \in W^{2,p}(\mathbb{R}^d)$:

$$\begin{aligned} ||Vf||_p &\leq ||\mathbb{1}_M Vf||_p + ||\mathbb{1}_{M^C} Vf||_p \leq C||f||_p + ||\mathbb{1}_{M^C} \mathcal{V}||_p ||f||_\infty \\ &\leq C||f||_p + \varepsilon ||f||_\infty. \end{aligned}$$

Application of Theorem 1

Last to check: Bound

For this let $\varepsilon > 0$ and $M \subseteq \mathbb{R}^d$, such that

$$\mathbb{1}_M \mathcal{V}(x) \leq C \quad \text{and} \quad ||\mathbb{1}_{M^c}\mathcal{V}||_p \leq \varepsilon.$$

With this, we can calculate for every $f \in W^{2,p}(\mathbb{R}^d)$:

$$\begin{aligned} ||Vf||_p &\leq ||\mathbb{1}_M Vf||_p + ||\mathbb{1}_{M^C} Vf||_p \leq C||f||_p + ||\mathbb{1}_{M^C}\mathcal{V}||_p||f||_\infty \\ &\leq C||f||_p + \varepsilon||f||_\infty. \end{aligned}$$

Use Sobolev-inequality, for $f \in W^{2,p}$:

$$||f||_\infty \leq ||f||_{W^{2,p}}$$

to get:

$$||Vf||_p \leq C||f||_p + \varepsilon||f||_{W^{2,p}}.$$

Application of Theorem 1

Claim: There is $D > 0$, such that $||f||_{W^{2,p}} \leq D||(-\triangle + 4e)f||_p$. With this, we would get

$$||Vf||_p \leq C||f||_p + \varepsilon D||(-\triangle + 4e)f||_p.$$

This would prove the bound.

Application of Theorem 1

Claim: There is $D > 0$, such that $||f||_{W^{2,p}} \leq D||(-\triangle + 4e)f||_p$. With this, we would get

$$||Vf||_p \leq C||f||_p + \varepsilon D||(-\triangle + 4e)f||_p.$$

This would prove the bound.
Prove:

We first prove for $f \in L^p(\mathbb{R}^d)$

$$||(-\triangle + 4e)^{-1}f||_{W^{2,p}} \leq C||f||_p.$$

Application of Theorem 1

Remember, that

$$||(-\triangle+4e)^{-1}f||_{W^{2,p}} = \sum_{|\alpha|\leq 2} ||D^{\alpha}(-\triangle+4e)^{-1}f||_p = \sum_{|\alpha|\leq 2} ||D^{\alpha}(Y_{4e}*f)||_p.$$

Application of Theorem 1

Remember, that

$$||(-\triangle+4e)^{-1}f||_{W^{2,p}} = \sum_{|\alpha|\leq 2} ||D^\alpha(-\triangle+4e)^{-1}f||_p = \sum_{|\alpha|\leq 2} ||D^\alpha(Y_{4e}*f)||_p.$$

Now we use $D^\alpha(g_1 * g_2) = (D^\alpha g_1) * g_2$, as well as Young's inequality $||g_1 * g_2||_p \leq ||g_1||_1||g_2||_p$ to get

$$\begin{aligned} ||(-\triangle+4e)^{-1}f||_{W^{2,p}} &= \sum_{|\alpha|\leq 2} ||D^\alpha(Y_{4e}*f)||_p \\ &= \sum_{|\alpha|\leq 2} ||(D^\alpha Y_{4e})*f||_p \\ &\leq \sum_{|\alpha|\leq 2} ||D^\alpha Y_{4e}||_1||f||_p \\ &= \left(\sum_{|\alpha|\leq 2} ||D^\alpha Y_{4e}||_1\right)||f||_p \end{aligned}$$

Application of Theorem 1

So we have shown

$$||(-\triangle + 4e)^{-1} f||_{W^{2,p}} \leq C||f||_p$$

for all $f \in L^p(\mathbb{R}^d)$.

Application of Theorem 1

So we have shown

$$||(-\triangle + 4e)^{-1} f||_{W^{2,p}} \leq C||f||_p$$

for all $f \in L^p(\mathbb{R}^d)$.

This shows $(-\triangle + 4e)^{-1} : L^p(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ is a bounded operator.

Application of Theorem 1

So we have shown

$$||(-\triangle+4e)^{-1}f||_{W^{2,p}} \leq C||f||_p$$

for all $f \in L^p(\mathbb{R}^d)$.

This shows $(-\triangle+4e)^{-1} : L^p(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ is a bounded operator.

Now we use that $(-\triangle+4e)^{-1}$ or equivalently $(-\triangle+4e)$ is a bijection, that is for every $f \in L^p(\mathbb{R}^d)$, there is a $f' \in W^{2,p}$ such that $f = (-\triangle+4e)f'$ and vice versa. Plugging this in, yields

$$||f'||_{W^{2,p}} \leq C||(-\triangle+4e)f'||_p$$

for all $f' \in W^{2,p}(\mathbb{R}^d)$, which was to show

□

Result and Corollaries:

To summarize:

We have proven that $H := (-\triangle + 4e + \mathcal{V}) : W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is closed and the generator of a contraction semigroup.

Result and Corollaries:

To summarize:

We have proven that $H := (-\triangle + 4e + \mathcal{V}) : W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is closed and the generator of a contraction semigroup.

Corollaries:

- Since H is closed and defined on all of $W^{2,p}(\mathbb{R}^d)$ it is bounded.

Result and Corollaries:

To summarize:

We have proven that $H := (-\triangle + 4e + \mathcal{V}) : W^{2,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is closed and the generator of a contraction semigroup.

Corollaries:

- Since H is closed and defined on all of $W^{2,p}(\mathbb{R}^d)$ it is bounded.
- From theorem (2), we know that for every $e, \lambda > 0$ the operator

$$H + \lambda I_d = -\triangle + 4e + \lambda + \mathcal{V}(x)$$

is surjective. Choosing for fixed $e_0 > 0$ $e = \frac{e_0}{2}$ and $\lambda = \frac{4e_0}{2}$, we get that for all $e_0 > 0$

$$-\triangle + 4\frac{e_0}{2} + 4\frac{e_0}{2} + \mathcal{V}(x) = -\triangle + 4e_0 + \mathcal{V}(x)$$

is surjective, hence H is surjective.

Corollaries:

For injectiveness, we are going to construct the inverse. Let for $f \in W^{2,p}(\mathbb{R}^d)$, let $g = Hf$ and $f(t) = e^{-Ht}f$. Since H is the generator of e^{-Ht} on $W^{2,p}(\mathbb{R}^d)$, we get

$$\partial_t f(t) = -Hf(t).$$

Corollaries:

For injectiveness, we are going to construct the inverse. Let for $f \in W^{2,p}(\mathbb{R}^d)$, let $g = Hf$ and $f(t) = e^{-Ht}f$. Since H is the generator of e^{-Ht} on $W^{2,p}(\mathbb{R}^d)$, we get

$$\partial_t f(t) = -Hf(t).$$

Integrating from 0 to t, we get:

$$\begin{aligned} f(t) - f &= \int_0^t \partial_t f(t) \text{ dt} = -\int_0^t Hf(t) \text{ dt} = -\int_0^t He^{-Ht}f \text{ dt} \\ &= -\int_0^t e^{-Ht}Hf \text{ dt} = -\int_0^t e^{-Ht}g \text{ dt} \end{aligned}.$$

Corollaries:

For injectiveness, we are going to construct the inverse. Let for $f \in W^{2,p}(\mathbb{R}^d)$, let $g = Hf$ and $f(t) = e^{-Ht}f$. Since H is the generator of e^{-Ht} on $W^{2,p}(\mathbb{R}^d)$, we get

$$\partial_t f(t) = -Hf(t).$$

Integrating from 0 to t, we get:

$$\begin{aligned} f(t) - f &= \int_0^t \partial_t f(t)\ \mathrm{dt} = -\int_0^t Hf(t)\ \mathrm{dt} = -\int_0^t He^{-Ht}f\ \mathrm{dt} \\ &= -\int_0^t e^{-Ht}Hf\ \mathrm{dt} = -\int_0^t e^{-Ht}g\ \mathrm{dt} \end{aligned}.$$

For $t \to \infty$, we get $f(t) \to 0$. Therefore in the limit

$$f = \int_0^\infty e^{-Ht}g\ \mathrm{dt}.$$

Having constructed the inverse, H is injective.

Corollaries:

Finally, we are going to prove, that H is positivity preserving, that is

$$u \geq 0 \Rightarrow Hu \geq 0.$$

Corollaries:

Finally, we are going to prove, that H is positivity preserving, that is

$$u \geq 0 \Rightarrow Hu \geq 0.$$

For this, use the Trotter product formula

$$e^{A+B} = \lim_{n \to \infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^{n},$$

with $A = (- \triangle + 4e)$ and $B = \mathcal{V}(x)$.

Corollaries:

Finally, we are going to prove, that H is positivity preserving, that is

$$u \geq 0 \Rightarrow Hu \geq 0.$$

For this, use the Trotter product formula

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n,$$

with $A = (- \triangle + 4e)$ and $B = \mathcal{V}(x)$. Both are positivity preserving.

Corollaries:

Finally, we are going to prove, that H is positivity preserving, that is

$$u \geq 0 \Rightarrow Hu \geq 0.$$

For this, use the Trotter product formula

$$e^{A+B} = \lim_{n \to \infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n,$$

with $A = (- \triangle + 4e)$ and $B = \mathcal{V}(x)$. Both are positivity preserving. Then there product is positivity preserving for all $n \in \mathbb{N}$.

Corollaries:

Finally, we are going to prove, that H is positivity preserving, that is

$$u \geq 0 \Rightarrow Hu \geq 0.$$

For this, use the Trotter product formula

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n,$$

with $A = (-\triangle + 4e)$ and $B = \mathcal{V}(x)$. Both are positivity preserving. Then there product is positivity preserving for all $n \in \mathbb{N}$. Then as the limit of positivity preserving operators $e^{(\triangle - 4e - \mathcal{V})t}$ is positivity preserving.

Corollaries:

Finally, we are going to prove, that H is positivity preserving, that is

$$u \geq 0 \Rightarrow Hu \geq 0.$$

For this, use the Trotter product formula

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n,$$

with $A = (-\triangle + 4e)$ and $B = \mathcal{V}(x)$. Both are positivity preserving. Then there product is positivity preserving for all $n \in \mathbb{N}$. Then as the limit of positivity preserving operators $e^{(\triangle - 4e - \mathcal{V})t}$ is positivity preserving. Then, for every function $u \geq 0$ in $L^p(\mathbb{R}^d)$, the integrand of

$$f = \int_0^\infty e^{-Ht} u \text{ dt}$$

is non-negative for all t. Since the integral over non-negative functions is positive, f is non-negative.

Summary

We have three ways of writing equation (1):

1. $$\big(-\triangle + 4e + \mathcal{V}(x)\big)u(x) = \mathcal{V}(x) + 2e\rho(e)(u * u)(x)$$

2. $$u(x) = Y_{4e} * \big(\mathcal{V}(1-u)\big)(x) + 2e\rho(e)(Y_{4e} * u * u)(x)$$

   with $Y_{4e} \geq 0$ and $\int Y_{4e} = \frac{1}{4e}$

3. $$u(x) = K_e\mathcal{V}(x) + 2e\rho(e)K_e(u * u)(x)$$

   with K_e being a bijection between $L^p(\mathbb{R}^d)$ and $W^{2,p}(\mathbb{R}^d)$ and positivity preserving.

For convenience, we would like to call them base equation 1, 2, 3 respectively in that order.

Defining (ρ_n) and (u_n)

Defining (ρ_n) and (u_n)

Let $e \in (0, \infty)$ be fixed. Then define recursively:

$$
\begin{aligned}
u_0(x) &:= 0 \\
u_n(x) &:= K_e \mathcal{V}(x) + 2e\rho_{n-1}(e) K_e(u_{n-1} * u_{n-1})(x)
\end{aligned}
$$

$$
\rho_n(e) := \frac{2e}{\int \big(1 - u_n(x)\big) \mathcal{V}(x) \; \mathrm{dx}}
$$

Overview

We are going to prove by induction:

- $u_n \in L^1(\mathbb{R}^d)$,
- $u_n \in L^p(\mathbb{R}^d)$,
- u_n is continuous,
- u_n vanishes at infinity
- $0 \leq u_n \leq 1$,
- ρ_n is well defined and positive.

Overview

We are going to prove by induction:

- $u_n \in L^1(\mathbb{R}^d)$,
- $u_n \in L^p(\mathbb{R}^d)$,
- u_n is continuous,
- u_n vanishes at infinity
- $0 \leq u_n \leq 1$,
- ρ_n is well defined and positive.

Base case n=0:

Overview

We are going to prove by induction:

- $u_n \in L^1(\mathbb{R}^d)$,
- $u_n \in L^p(\mathbb{R}^d)$,
- u_n is continuous,
- u_n vanishes at infinity
- $0 \leq u_n \leq 1$,
- ρ_n is well defined and positive.

Base case n=0:

$u_0 = 0$ satisfies all the above properties. Furthermore

$$\rho_0 = \frac{2e}{\int \mathcal{V}(x)\ \mathrm{dx}} = \frac{2e}{||\mathcal{V}||_1} > 0.$$

Induction step

Induction step $n \in \mathbb{N}$

We look at the defining equation

$$u_n(x) := K_e \mathcal{V}(x) + 2e\rho_{n-1}(e) K_e(u_{n-1} * u_{n-1})(x).$$

By assumption $\mathcal{V}, u_{n-1} \in L^p(\mathbb{R}^d)$, therefore $u_n \in W^{2,p}(\mathbb{R}^d)$. It follows:

- $u_n \in L^p(\mathbb{R}^d)$ ✓
- $u_n \in L^1(\mathbb{R}^d)$ ✓
  (Since the prove of $K_e : L^p(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ only used boundedness of $W^{2,p}(\mathbb{R}^d)$, which holds for $W^{2,1}(\mathbb{R}^d)$.)
- u_n is continuous ✓
  (Follows again, from the embedding $W^{k,p}(\Omega) \subseteq C(\Omega)$ for $kp \geq n$.)
- u_n vanishes at infinity ✓

Induction step

Induction step $n \in \mathbb{N}$

We look at the defining equation

$$u_n(x) := K_e\mathcal{V}(x) + 2e\rho_{n-1}(e)K_e(u_{n-1} * u_{n-1})(x).$$

By assumption $\mathcal{V}, u_{n-1} \in L^p(\mathbb{R}^d)$, therefore $u_n \in W^{2,p}(\mathbb{R}^d)$. It follows:

- $u_n \in L^p(\mathbb{R}^d)$ ✓
- $u_n \in L^1(\mathbb{R}^d)$ ✓
  (Since the prove of $K_e : L^p(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ only used boundedness of $W^{2,p}(\mathbb{R}^d)$, which holds for $W^{2,1}(\mathbb{R}^d)$.)
- u_n is continuous ✓
  (Follows again, from the embedding $W^{k,p}(\Omega) \subseteq C(\Omega)$ for $kp \geq n$.)
- u_n vanishes at infinity ✓

Furthermore, since K_e preserves positivity

- $0 \leq u_n$ ✓

Induction step $n \in \mathbb{N}$

We look at the defining equation

$$u_n(x) := K_e\mathcal{V}(x) + 2e\rho_{n-1}(e)K_e(u_{n-1} * u_{n-1})(x).$$

By assumption $\mathcal{V}, u_{n-1} \in L^p(\mathbb{R}^d)$, therefore $u_n \in W^{2,p}(\mathbb{R}^d)$. It follows:

- $u_n \in L^p(\mathbb{R}^d)$ ✓
- $u_n \in L^1(\mathbb{R}^d)$ ✓
  (Since the prove of $K_e : L^p(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ only used boundedness of $W^{2,p}(\mathbb{R}^d)$, which holds for $W^{2,1}(\mathbb{R}^d)$.)
- u_n is continuous ✓
  (Follows again, from the embedding $W^{k,p}(\Omega) \subseteq C(\Omega)$ for $kp \geq n$.)
- u_n vanishes at infinity ✓

Furthermore, since K_e preserves positivity

- $0 \leq u_n$ ✓

We need to check:

$$u \leq 1 \qquad \text{and} \qquad \rho_n > 0$$

Intermezzo: Lemma

Lemma

For all $n \in \mathbb{N}$, we have

- $u_n \geq u_{n-1}$
- $\rho_n \geq \rho_{n-1}$
- $\int u_n(x)\ \mathrm{dx} \leq \frac{\int \mathcal{V}(x)\left(1-u(x)\right)\mathrm{dx}}{2e} \Rightarrow \rho_n ||u_n||_1 \leq 1.$

Intermezzo: Lemma

Lemma

For all $n \in \mathbb{N}$, we have

- $u_n \geq u_{n-1}$
- $\rho_n \geq \rho_{n-1}$
- $\int u_n(x)\,\mathrm{dx} \leq \frac{\int \mathcal{V}(x)\big(1-u(x)\big)\,\mathrm{dx}}{2e} \Rightarrow \rho_n ||u_n||_1 \leq 1.$

Prove (by induction):

For the base case $n = 1$, we have (using K_e preserves positivity)

$$u_1(x) = K_e \mathcal{V}(x) \geq 0 = u_0(x).$$

Intermezzo: Lemma

Prove (by induction):

For the base case $n = 1$, we have (using K_e preserves positivity and is bijective)

$$u_1(x) = K_e \mathcal{V}(x) > 0 = u_0(x).$$

Intermezzo: Lemma

Prove (by induction):

For the base case $n = 1$, we have (using K_e preserves positivity and is bijective)

$$u_1(x) = K_e \mathcal{V}(x) > 0 = u_0(x).$$

For $n = 1$ base equation 2 reads

$$u_1(x) = Y_{4e} * \big(\mathcal{V}(1 - u_1)\big)(x),$$

Prove (by induction):

For the base case $n = 1$, we have (using K_e preserves positivity and is bijective)

$$u_1(x) = K_e \mathcal{V}(x) > 0 = u_0(x).$$

For $n = 1$ base equation 2 reads

$$u_1(x) = Y_{4e} * \big(\mathcal{V}(1 - u_1)\big)(x),$$

which integrated gives

$$\int u_1(x) \, \mathrm{dx} = \frac{1}{4e} \int \mathcal{V}(x)\big(1 - u_1(x)\big) \, \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\big(1 - u_1(x)\big) \, \mathrm{dx}.$$

This shows

$$0 \leq \frac{1}{2e} \int \mathcal{V}(x)\big(1 - u_1(x)\big) \, \mathrm{dx}.$$

Intermezzo: Lemma

Now it follows:

$$\rho_1 = \frac{2e}{\int \mathcal{V}(x)\big(1 - u_1(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\ \mathrm{dx}} = \rho_0$$

where the denominator is not zero, because either

- $u_1 = 0$ almost everywhere $\Rightarrow \int \mathcal{V}(x)\big(1 - u_1(x)\big)\ \mathrm{dx} = \int \mathcal{V}(x)\ \mathrm{dx} > 0$
- $\int u_1(x)\ \mathrm{dx} > 0 \Rightarrow \int \mathcal{V}(x)\big(1 - u_1(x)\big)\ \mathrm{dx} \geq \int u_1(x)\ \mathrm{dx} > 0$ by the bound on the slide before.

Intermezzo: Lemma

Prove (by induction):

Now let $n \geq 2$. Then by induction hypothesis

$$\begin{aligned} u_n &= K_e\mathcal{V} + 2\rho_{n-1}(e)K_e(u_{n-1} * u_{n-1})(x) \\ &\geq K_e\mathcal{V} + 2\rho_{n-2}(e)K_e(u_{n-2} * u_{n-2})(x) = u_{n-1}(x). \end{aligned}$$

Prove (by induction):

Now let $n \geq 2$. Then by induction hypothesis

$$\begin{aligned} u_n &= K_e \mathcal{V} + 2\rho_{n-1}(e) K_e(u_{n-1} * u_{n-1})(x) \\ &\geq K_e \mathcal{V} + 2\rho_{n-2}(e) K_e(u_{n-2} * u_{n-2})(x) = u_{n-1}(x). \end{aligned}$$

Integrating base equation 2, we get

$$\begin{aligned} \int u_n &= \frac{1}{4e} \int \mathcal{V}(x)\big(1 - u_n(x)\big) \, \mathrm{dx} + \frac{\rho_{n-1}(e)}{2} \left(\int u_{n-1}(x) \, \mathrm{dx} \right)^2 \\ &\leq \frac{1}{4e} \int \mathcal{V}(x)\big(1 - u_n(x)\big) \, \mathrm{dx} + \frac{1}{2} \left(\int u_{n-1}(x) \, \mathrm{dx} \right) \end{aligned}$$

Rearranging gives:

$$\frac{1}{2e} \int \mathcal{V}(x)\big(1 - u_n(x)\big) \, \mathrm{dx} \geq 2 \int u_n \, \mathrm{dx} - \int u_{n-1}(x) \, \mathrm{dx} \geq \int u_n \, \mathrm{dx}$$

Intermezzo: Lemma

Prove (by induction):

$$\rho_n = \frac{2e}{\int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\big(1 - u_{n-1}(x)\big)\ \mathrm{dx}} = \rho_{n-1}$$

where once again the denominator is not zero, because either

- $u_n = 0$ almost everywhere $\Rightarrow \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} = \int \mathcal{V}(x)\ \mathrm{dx} > 0$
- $\int u_n(x)\ \mathrm{dx} > 0 \Rightarrow \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} \geq \int u_n(x)\ \mathrm{dx} > 0.$

Intermezzo: Lemma

Prove (by induction):

$$\rho_n = \frac{2e}{\int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\big(1 - u_{n-1}(x)\big)\ \mathrm{dx}} = \rho_{n-1}$$

where once again the denominator is not zero, because either

- $u_n = 0$ almost everywhere $\Rightarrow \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} = \int \mathcal{V}(x)\ \mathrm{dx} > 0$
- $\int u_n(x)\ \mathrm{dx} > 0 \Rightarrow \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} \geq \int u_n(x)\ \mathrm{dx} > 0.$

□

Intermezzo: Lemma

Prove (by induction):

$$\rho_n = \frac{2e}{\int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\big(1 - u_{n-1}(x)\big)\ \mathrm{dx}} = \rho_{n-1}$$

where once again the denominator is not zero, because either

- $u_n = 0$ almost everywhere $\Rightarrow \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} = \int \mathcal{V}(x)\ \mathrm{dx} > 0$
- $\int u_n(x)\ \mathrm{dx} > 0 \Rightarrow \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} \geq \int u_n(x)\ \mathrm{dx} > 0.$

□

This shows

- ρ_n is well-defined and positive ✓

Induction step

To prove $u_n \leq 1$ assume the opposite and define

$$A := \{x \in \mathbb{R}^d | u_n(x) > 1\}.$$

Induction step

To prove $u_n \leq 1$ assume the opposite and define

$$A := \{x \in \mathbb{R}^d | u_n(x) > 1\}.$$

u_n bounded and continuous $\Rightarrow$ u_n has a maximum x_0.

Induction step

To prove $u_n \leq 1$ assume the opposite and define

$$A := \{x \in \mathbb{R}^d | u_n(x) > 1\}.$$

u_n bounded and continuous $\Rightarrow$ u_n has a maximum x_0.

A non-empty $\Rightarrow$ $x_0 \in A$.

Induction step

To prove $u_n \leq 1$ assume the opposite and define

$$A := \{x \in \mathbb{R}^d | u_n(x) > 1\}.$$

u_n bounded and continuous $\Rightarrow$ u_n has a maximum x_0.

A non-empty $\Rightarrow x_0 \in A$.
Rearranging the defining equation for u_n, we get:

$$\begin{aligned}\triangle u_n(x) &= \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}\big(u_{n-1} * u_{n-1}\big)(x) \\ &\geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}||u_{n-1} * u_{n-1}||_\infty\end{aligned}$$

Induction step

To prove $u_n \leq 1$ assume the opposite and define

$$A := \{x \in \mathbb{R}^d | u_n(x) > 1\}.$$

u_n bounded and continuous $\Rightarrow$ u_n has a maximum x_0.

A non-empty $\Rightarrow$ $x_0 \in A$.
Rearranging the defining equation for u_n, we get:

$$\begin{aligned} \triangle u_n(x) &= \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}\big(u_{n-1} * u_{n-1}\big)(x) \\ &\geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}||u_{n-1} * u_{n-1}||_\infty \end{aligned}$$

We use $||u_{n-1} * u_{n-1}||_\infty \leq ||u_{n-1}||_1 ||u_{n-1}||_\infty$ and use $||u_{n-1}||_\infty \leq 1$:

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}||u_{n-1}||_1.$$

Induction step

To prove $u_n \leq 1$ assume the opposite and define

$$A := \{x \in \mathbb{R}^d | u_n(x) > 1\}.$$

u_n bounded and continuous $\Rightarrow$ u_n has a maximum x_0.

A non-empty $\Rightarrow x_0 \in A$.
Rearranging the defining equation for u_n, we get:

$$\begin{aligned}\triangle u_n(x) &= \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}\big(u_{n-1} * u_{n-1}\big)(x) \\ &\geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}||u_{n-1} * u_{n-1}||_\infty\end{aligned}$$

We use $||u_{n-1} * u_{n-1}||_\infty \leq ||u_{n-1}||_1 ||u_{n-1}||_\infty$ and use $||u_{n-1}||_\infty \leq 1$:

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e\rho_{n-1}||u_{n-1}||_1.$$

Now we use $\rho_{n-1}||u_{n-1} \leq 1$:

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e.$$

Induction step

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e.$$

Induction step

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e.$$

Assume $x \in A$. Then $u_n(x) > 1$.

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 2e > 0.$$

Induction step

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e.$$

Assume $x \in A$. Then $u_n(x) > 1$.

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 2e > 0.$$

But for local maxima $\triangle u_n(x) \leq 0$. We have a contradiction.

Induction step

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 4eu_n(x) - 2e.$$

Assume $x \in A$. Then $u_n(x) > 1$.

$$\triangle u_n(x) \geq \mathcal{V}\big(u_n(x) - 1\big) + 2e > 0.$$

But for local maxima $\triangle u_n(x) \leq 0$. We have a contradiction.

$\Rightarrow A$ is empty $\Rightarrow u_n \leq 1$ ✓

Proving the limits of (ρ_n) and (u_n) solve (1)

Existence

To prove:

- (ρ_n) and (u_n) monotonic increasing

Existence

To prove:

- (ρ_n) and (u_n) monotonic increasing ✓

Existence

To prove:

- (ρ_n) and (u_n) monotonic increasing ✓
- (ρ_n) and (u_n) are bounded

Existence

To prove:

- (ρ_n) and (u_n) monotonic increasing ✓
- (ρ_n) and (u_n) are bounded

For (u_n) $f(x) = 1$ is an upper bound.

Existence

To prove:

- (ρ_n) and (u_n) monotonic increasing ✓
- (ρ_n) and (u_n) are bounded

For (u_n) $f(x) = 1$ is an upper bound.

For (ρ_n) use $\rho_n ||u_n||_1 \leq 1$ and $||u_n||_1 \geq ||u_1||$:

$$\rho_n \leq \frac{1}{||u_n||} \leq \frac{1}{||u_1||}.$$

Existence

To prove:

- (ρ_n) and (u_n) monotonic increasing ✓
- (ρ_n) and (u_n) are bounded

For (u_n) $f(x) = 1$ is an upper bound.

For (ρ_n) use $\rho_n ||u_n||_1 \leq 1$ and $||u_n||_1 \geq ||u_1||$:

$$\rho_n \leq \frac{1}{||u_n||} \leq \frac{1}{||u_1||}.$$

The limits

$$\rho := \lim_{n\to\infty} \rho_n \qquad \text{and} \qquad u(x) := \lim_{n\to\infty} u_n(x)$$

exist.

Integrability

As $|u_n| \leq 1$, we get

$$\int u_n(x)\ \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\ \mathrm{dx}.$$

Integrability

As $|u_n| \leq 1$, we get

$$\int u_n(x)\ \mathrm{dx} \leq \frac{1}{2e}\int \mathcal{V}(x)\big(1-u_n(x)\big)\ \mathrm{dx} \leq \frac{1}{2e}\int \mathcal{V}(x)\ \mathrm{dx}.$$

So (u_n) is

- almost everywhere converging
- integrable
- has bounded 1-norm

Integrability

As $|u_n| \leq 1$, we get

$$\int u_n(x)\ \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\ \mathrm{dx}.$$

So (u_n) is

- almost everywhere converging
- integrable
- has bounded 1-norm

By Fatou's lemma: $u \in L^1(\mathbb{R}^d)$.

Integrability

As $|u_n| \leq 1$, we get

$$\int u_n(x)\ \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\big(1 - u_n(x)\big)\ \mathrm{dx} \leq \frac{1}{2e} \int \mathcal{V}(x)\ \mathrm{dx}.$$

So (u_n) is

- almost everywhere converging
- integrable
- has bounded 1-norm

By Fatou's lemma: $u \in L^1(\mathbb{R}^d)$.

Then by dominated convergence, as u is upper bound for all u_n:

$$\lim_{n\to\infty} \int u_n(x)\ \mathrm{dx} = \int u(x)\ \mathrm{dx} \quad \Leftrightarrow \quad \lim_{n\to\infty} ||u - u_n||_1 = 0.$$

Integrability

As $0 \leq u(x) \leq 1$

$$||u||_p^p = \int u^p(x) \text{ dx} \leq \int u(x) \text{ dx} = ||u||_1,$$

therefore $u \in L^p(\mathbb{R}^d)$.

Integrability

As $0 \leq u(x) \leq 1$

$$||u||_p^p = \int u^p(x) \text{ dx} \leq \int u(x) \text{ dx} = ||u||_1,$$

therefore $u \in L^p(\mathbb{R}^d)$.

As $1 \geq u(x) \geq u_n \geq 0$, we have $u(x) - u_n(x) \leq 1$, therefore

$$||u - u_n||_p^p = \int \left|u(x) - u_n(x)\right|^p \text{ dx} \leq \int \left|u(x) - u_n(x)\right| \text{ dx} = ||u - u_n||_1,$$

proving

$$\lim_{n\to\infty} ||u - u_n||_p = 0.$$

Solution-property

With this:

$$\begin{aligned}||u * u - u_n * u_n||_p &= ||u * u - u * u_n + u * u_n - u_n * u_n||_p \\ &\leq ||u * (u - u_n)||_p + ||(u - u_n) * u_n||_p \\ &\leq ||u||_1 ||u - u_n||_p + ||u_n||_1 ||u - u_n||_p\end{aligned}$$

This shows

$$\lim_{n\to\infty} ||u * u - u_n * u_n|| = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} u_n * u_n = u * u \text{ (in } p\text{-Norm).}$$

Solution-property

With this:

$$\begin{aligned} ||u * u - u_n * u_n||_p &= ||u * u - u * u_n + u * u_n - u_n * u_n||_p \\ &\leq ||u * (u - u_n)||_p + ||(u - u_n) * u_n||_p \\ &\leq ||u||_1 ||u - u_n||_p + ||u_n||_1 ||u - u_n||_p \end{aligned}$$

This shows

$$\lim_{n \to \infty} ||u * u - u_n * u_n|| = 0 \quad \Leftrightarrow \quad \lim_{n \to \infty} u_n * u_n = u * u \text{ (in } p\text{-Norm).}$$

Now taking the limit of the defining equation

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(K_e \mathcal{V} + 2e\rho_{n-1} K_e (u_{n-1} * u_{n-1}) \right)$$

we get

$$u = K_e \mathcal{V} + 2e\rho K_e (u * u).$$

Solution-property

Left to check:

$$\rho(e) = \frac{2e}{\int \left(1 - u(x)\right) \mathcal{V}(x) \, \mathrm{dx}}.$$

Solution-property

Left to check:

$$\rho(e) = \frac{2e}{\int \big(1 - u(x)\big)\mathcal{V}(x)\,\mathrm{dx}}.$$

For this, we take the limit and use dominated convergence

$$\rho = \lim_{n\to\infty} \frac{2e}{\int \mathcal{V}(1-u_n)\,\mathrm{dx}} = \frac{2e}{\lim_{n\to\infty}\int \mathcal{V}(1-u_n)\,\mathrm{dx}} = \frac{2e}{\int \mathcal{V}(1-u)\,\mathrm{dx}}$$

Proving ρ is continuous and has the limit properties

Continuity of $\rho_n(e)$ and $u_n(x, e)$

Claim:

$\rho(e)$ and $u_n(x, e)$ are continuous in e for all $n \in \mathbb{N}_0$.

Continuity of $\rho_n(e)$ and $u_n(x, e)$

Claim:

$\rho(e)$ and $u_n(x, e)$ are continuous in e for all $n \in \mathbb{N}_0$.

Prove (by induction):

Continuity of $\rho_n(e)$ and $u_n(x,e)$

Claim:

$\rho(e)$ and $u_n(x,e)$ are continuous in e for all $n \in \mathbb{N}_0$.

Prove (by induction):

For $n = 0$: $u_n(x,e) = 0$ and $\rho_0(e) = \frac{2e}{\int \mathcal{V}(x)\mathrm{dx}}$ are continuous.

Continuity of $\rho_n(e)$ and $u_n(x, e)$

Claim:

$\rho(e)$ and $u_n(x, e)$ are continuous in e for all $n \in \mathbb{N}_0$.

Prove (by induction):

For $n = 0$: $u_n(x, e) = 0$ and $\rho_0(e) = \frac{2e}{\int \mathcal{V}(x)\mathrm{dx}}$ are continuous.

For $n \in \mathbb{N}$ we observe K_e is continuous in e, as

$$e \mapsto -\triangle + 4e + \mathcal{V}(x) \overset{(\cdot)^{-1}}{\mapsto} K_e,$$

such that $e \mapsto K_e f$ is continuous for $f \in L^p(\mathbb{R}^d)$.

Continuity of $\rho_n(e)$ and $u_n(x,e)$

Claim:

$\rho(e)$ and $u_n(x,e)$ are continuous in e for all $n \in \mathbb{N}_0$.

Prove (by induction):

For $n = 0$: $u_n(x,e) = 0$ and $\rho_0(e) = \frac{2e}{\int \mathcal{V}(x)\mathrm{dx}}$ are continuous.

For $n \in \mathbb{N}$ we observe K_e is continuous in e, as

$$e \mapsto -\triangle + 4e + \mathcal{V}(x) \overset{(\cdot)^{-1}}{\mapsto} K_e,$$

such that $e \mapsto K_e f$ is continuous for $f \in L^p(\mathbb{R}^d)$.

It then follows, that $u_n(x,e)$ is continuous in e as

$$u_n(x,e) = K_e\mathcal{V}(x) + 2e\rho_{n-1}(e)\big(K_e u_{n-1} * u_{n-1}\big)(x,e).$$

Continuity of $\rho_n(e)$ and $u_n(x,e)$

Claim:

$\rho(e)$ and $u_n(x,e)$ are continuous in e for all $n \in \mathbb{N}_0$.

Prove (by induction):

For $n = 0$: $u_n(x,e) = 0$ and $\rho_0(e) = \frac{2e}{\int \mathcal{V}(x)\mathrm{dx}}$ are continuous.

For $n \in \mathbb{N}$ we observe K_e is continuous in e, as

$$e \mapsto -\triangle + 4e + \mathcal{V}(x) \overset{(\cdot)^{-1}}{\mapsto} K_e,$$

such that $e \mapsto K_e f$ is continuous for $f \in L^p(\mathbb{R}^d)$.

It then follows, that $u_n(x,e)$ is continuous in e as

$$u_n(x,e) = K_e\mathcal{V}(x) + 2e\rho_{n-1}(e)\big(K_e u_{n-1} * u_{n-1}\big)(x,e).$$

This implies (with dominated convergence)

$$\rho_n(e) = \frac{2e}{\int \mathcal{V}(x)\big(1 - u_n(x,e)\big)}\,\mathrm{dx}$$

is continuous in e.

Uniform convergence of $\rho_n(e)$

Claim:

$\rho(e)$ converges uniformly toward $\rho(e)$.

Uniform convergence of $\rho_n(e)$

Claim:

$\rho(e)$ converges uniformly toward $\rho(e)$.

What to prove:

$$\forall e \in (0,\infty): \qquad \left|\rho(e) - \rho_n(e)\right| \leq \frac{C}{n}$$

Uniform convergence of $\rho_n(e)$

Claim:

$\rho(e)$ converges uniformly toward $\rho(e)$.

What to prove:

$$\forall e \in (0,\infty): \quad \left|\rho(e) - \rho_n(e)\right| \leq \frac{C}{n}$$

Modification 1

$$\forall e \in (0,\infty): \quad \left|\frac{1}{\rho(e)} - \frac{1}{\rho_n(e)}\right| \leq \frac{C}{n}$$

Uniform convergence of $\rho_n(e)$

Claim:

$\rho(e)$ converges uniformly toward $\rho(e)$.

What to prove:

$$\forall e \in (0, \infty): \quad \left|\rho(e) - \rho_n(e)\right| \leq \frac{C}{n}$$

Modification 1

$$\forall e \in (0, \infty): \quad \left|\frac{1}{\rho(e)} - \frac{1}{\rho_n(e)}\right| \leq \frac{C}{n}$$

Modification 2

$$\forall e \in (0, \infty): \quad \left|a_n(e) - \frac{1}{\rho_n(e)}\right| \leq \frac{C}{n}$$

for $(a_n) \to \frac{1}{\rho}$ monotonically increasing.

Uniform convergence of $\rho_n(e)$

Claim:

$\rho(e)$ converges uniformly toward $\rho(e)$.

What to prove:

$$\forall e \in (0, \infty) : \quad \left|\rho(e) - \rho_n(e)\right| \leq \frac{C}{n}$$

Modification 1

$$\forall e \in (0, \infty) : \quad \left|\frac{1}{\rho(e)} - \frac{1}{\rho_n(e)}\right| \leq \frac{C}{n}$$

Modification 2

$$\forall e \in (0, \infty) : \quad \left|a_n(e) - \frac{1}{\rho_n(e)}\right| \leq \frac{C}{n}$$

for $(a_n) \to \frac{1}{\rho}$ monotonically increasing.

Uniform convergence of $\rho_n(e)$

Modification 2

$$\forall e \in (0,\infty): \quad \left| a_n(e) - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n}$$

for $(a_n) \to \frac{1}{\rho}$ monotonically increasing. This holds because of

$$\forall e \in (0,\infty): \left| a_n(e) - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n} \Rightarrow \forall e \in (0,\infty): \left| \frac{1}{\rho} - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n}$$

as $a_n \leq \frac{1}{\rho}$ and $\frac{1}{\rho} \geq \frac{1}{\rho_n}$.

Uniform convergence of $\rho_n(e)$

Modification 2

$$\forall e \in (0,\infty): \quad \left| a_n(e) - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n}$$

for $(a_n) \to \frac{1}{\rho}$ monotonically increasing. This holds because of

$$\forall e \in (0,\infty): \ \left| a_n(e) - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n} \Rightarrow \forall e \in (0,\infty): \ \left| \frac{1}{\rho} - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n}$$

as $a_n \leq \frac{1}{\rho}$ and $\frac{1}{\rho} \geq \frac{1}{\rho_n}$.

Modification 3

$$\forall e \in [e_1, e_2]: \quad \left| a_n(e) - \frac{1}{\rho_n(e)} \right| \leq \frac{C}{n}$$

Uniform convergence of $\rho_n(e)$

Prove:

Define

$$a_n(e) = \int u_n(x,e)\ \mathrm{dx} \qquad \text{and} \qquad \frac{1}{\rho_n} = b_n = \frac{1}{2e}\int \mathcal{V}(x)\big(1 - u_n(x,e)\big)\mathrm{dx}.$$

Uniform convergence of $\rho_n(e)$

Prove:

Define

$$a_n(e) = \int u_n(x,e)\ \mathrm{dx} \qquad \text{and} \qquad \frac{1}{\rho_n} = b_n = \frac{1}{2e}\int \mathcal{V}(x)\big(1 - u_n(x,e)\big)\mathrm{dx}.$$

(a_n) has the following properties:

- $a_n(e) = \int u_n(x,e)\ \mathrm{dx}$ is continuous in e,
- $\int u_n(x,e)\ \mathrm{dx} \geq \int u_{n-1}(x,e)\ \mathrm{dx} \Rightarrow a_n \geq a_{n-1}$
- $\int u_n(x,e)\ \mathrm{dx} \leq \int u(x,e)\ \mathrm{dx} \leq \frac{1}{\rho} \Rightarrow a_n \leq \frac{1}{\rho}$
- $\int u_n(x,e)\ \mathrm{dx} \overset{n\to\infty}{\to} \int u(x,e)\ \mathrm{dx} = \frac{1}{\rho} \Rightarrow a_n \overset{n\to\infty}{\to} \frac{1}{\rho}$.
  (Equality in 4 will be proven on the next slide!)

Uniform convergence of $\rho_n(e)$

Claim:

If u, ρ are (integrable) solutions to equation (1), then

$$\int u(x, e)\ \mathrm{dx} = \frac{1}{\rho}$$

Uniform convergence of $\rho_n(e)$

Claim:

If u, ρ are (integrable) solutions to equation (1), then

$$\int u(x,e)\ \mathrm{dx} = \frac{1}{\rho}$$

Prove:

We start with base equation 2:

$$u(x) = Y_{4e} * \big(\mathcal{V}(1-u)\big)(x) + 2e\rho(e)(Y_{4e} * u * u)(x)$$

and integrate

$$\int u(x)\ \mathrm{dx} = \frac{1}{4e}\int \mathcal{V}(x)(1-u(x))\ \mathrm{dx} + \frac{\rho}{2}\left(\int u(x)\ \mathrm{dx}\right)^2,$$

where we have used $\int u * u \mathrm{dx} = \left(\int u \mathrm{dx}\right)^2$.

Uniform convergence of $\rho_n(e)$

Using

$$\rho = \frac{2e}{\int \big(1 - u(x)\big)\mathcal{V}(x)\ \mathrm{dx}}$$

we get

$$\int u(x)\ \mathrm{dx} = \frac{1}{2\rho} + \frac{\rho}{2}\left(\int u(x)\ \mathrm{dx}\right)^2.$$

Uniform convergence of $\rho_n(e)$

Using

$$\rho = \frac{2e}{\int \big(1 - u(x)\big)\mathcal{V}(x)\ \mathrm{dx}}$$

we get

$$\int u(x)\ \mathrm{dx} = \frac{1}{2\rho} + \frac{\rho}{2}\left(\int u(x)\ \mathrm{dx}\right)^2.$$

Rearranging

$$\rho^2\left(\int u(x)\ \mathrm{dx}\right)^2 - 2\rho\int u(x)\ \mathrm{dx} + 1 = \left(\rho\int u(x)\ \mathrm{dx} - 1\right)^2 = 0.$$

Uniform convergence of $\rho_n(e)$

Using

$$\rho = \frac{2e}{\int (1 - u(x))\mathcal{V}(x)\ \mathrm{dx}}$$

we get

$$\int u(x)\ \mathrm{dx} = \frac{1}{2\rho} + \frac{\rho}{2}\left(\int u(x)\ \mathrm{dx}\right)^2.$$

Rearranging

$$\rho^2\left(\int u(x)\ \mathrm{dx}\right)^2 - 2\rho\int u(x)\ \mathrm{dx} + 1 = \left(\rho\int u(x)\ \mathrm{dx} - 1\right)^2 = 0.$$

This proves

$$\rho\int u(x)\ \mathrm{dx} - 1 = 0 \Rightarrow \frac{1}{\rho} = \int u(x)\ \mathrm{dx}.$$

□

Uniform convergence of $\rho_n(e)$

(a_n) has the following properties:

- $a_n(e) = \int u_n(x,e)\,\mathrm{dx}$ is continuous in e,
- $\int u_n(x,e)\,\mathrm{dx} \geq \int u_{n-1}(x,e)\,\mathrm{dx} \Rightarrow a_n \geq a_{n-1}$
- $\int u_n(x,e)\,\mathrm{dx} \leq \int u(x,e)\,\mathrm{dx} \leq \frac{1}{\rho} \Rightarrow a_n \leq \frac{1}{\rho}$
- $\int u_n(x,e)\,\mathrm{dx} \overset{n\to\infty}{\to} \int u(x,e)\,\mathrm{dx} = \frac{1}{\rho} \Rightarrow a_n \overset{n\to\infty}{\to} \frac{1}{\rho}$.

Uniform convergence of $\rho_n(e)$

(a_n) has the following properties:

- $a_n(e) = \int u_n(x, e)\ \mathrm{dx}$ is continuous in e,
- $\int u_n(x, e)\ \mathrm{dx} \geq \int u_{n-1}(x, e)\ \mathrm{dx} \Rightarrow a_n \geq a_{n-1}$
- $\int u_n(x, e)\ \mathrm{dx} \leq \int u(x, e)\ \mathrm{dx} \leq \frac{1}{\rho} \Rightarrow a_n \leq \frac{1}{\rho}$
- $\int u_n(x, e)\ \mathrm{dx} \overset{n\to\infty}{\to} \int u(x, e)\ \mathrm{dx} = \frac{1}{\rho} \Rightarrow a_n \overset{n\to\infty}{\to} \frac{1}{\rho}$.

From the fact that $b_n = \frac{1}{\rho_n}$, we immediately see that

- b_n is continuous in e,
- b_n is monotonic decreasing
- b_n converges towards $\frac{1}{\rho}$ from above.

Uniform convergence of $\rho_n(e)$

For the bound, we start with the integrated version of base equation 2:

$$\int u_n(x)\ \mathrm{dx} = \frac{1}{4e}\int \mathcal{V}(x)\big(1-u_n(x)\big)\ \mathrm{dx} + \frac{\rho_{n-1}}{2}\left(\int u_{n-1}(x)\ \mathrm{dx}\right)^2$$

and replace a_n and b_n:

$$2a_n(e) = b_n(e) + \frac{1}{b_{n-1}(e)}a_{n-1}^2(e).$$

Uniform convergence of $\rho_n(e)$

$$2a_n(e) = b_n(e) + \frac{1}{b_{n-1}(e)} a_{n-1}^2(e).$$

Using this, we arrive at

$$\frac{1}{b_n(e)}\big(a_n(e) - b_n(e)\big)^2 = \frac{a_n^2(e)}{b_n(e)} - 2a_n(e) + b_n(e) = \frac{a_n^2(e)}{b_n(e)} - \frac{a_{n-1}^2(e)}{b_{n-1}(e)}.$$

Uniform convergence of $\rho_n(e)$

$$2a_n(e) = b_n(e) + \frac{1}{b_{n-1}(e)} a_{n-1}^2(e).$$

Using this, we arrive at

$$\frac{1}{b_n(e)}\big(a_n(e) - b_n(e)\big)^2 = \frac{a_n^2(e)}{b_n(e)} - 2a_n(e) + b_n(e) = \frac{a_n^2(e)}{b_n(e)} - \frac{a_{n-1}^2(e)}{b_{n-1}(e)}.$$

Summing over all n:

$$\sum_{n\in\mathbb{N}} \frac{1}{b_n(e)}\big(a_n(e) - b_n(e)\big)^2 = \frac{1}{\rho(e)}.$$

Uniform convergence of $\rho_n(e)$

$$2a_n(e) = b_n(e) + \frac{1}{b_{n-1}(e)} a_{n-1}^2(e).$$

Using this, we arrive at

$$\frac{1}{b_n(e)} \left(a_n(e) - b_n(e)\right)^2 = \frac{a_n^2(e)}{b_n(e)} - 2a_n(e) + b_n(e) = \frac{a_n^2(e)}{b_n(e)} - \frac{a_{n-1}^2(e)}{b_{n-1}(e)}.$$

Summing over all n:

$$\sum_{n\in\mathbb{N}} \frac{1}{b_n(e)} \left(a_n(e) - b_n(e)\right)^2 = \frac{1}{\rho(e)}.$$

Using $b_n < b_1$:

$$\sum_{n\in\mathbb{N}} \frac{1}{b_1(e)} \left(a_n(e) - b_n(e)\right)^2 \leq \frac{1}{\rho(e)}.$$

Uniform convergence of $\rho_n(e)$

$$\sum_{n\in\mathbb{N}} \frac{1}{b_1(e)}\big(a_n(e) - b_n(e)\big)^2 \leq \frac{1}{\rho(e)}.$$

Uniform convergence of $\rho_n(e)$

$$\sum_{n\in\mathbb{N}} \frac{1}{b_1(e)} \big(a_n(e) - b_n(e)\big)^2 \leq \frac{1}{\rho(e)}.$$

Rearranging:

$$\sum_{n\in\mathbb{N}} \big(a_n(e) - b_n(e)\big)^2 \leq \frac{b_1(e)}{\rho(e)} = \frac{\int \mathcal{V}(x)\big(1 - u(x,e)\big)\ \mathrm{dx}}{\int K_e \mathcal{V}(x)\ \mathrm{dx}} \leq \frac{\int \mathcal{V}(x)\ \mathrm{dx}}{\int K_e \mathcal{V}(x)\ \mathrm{dx}}.$$

Uniform convergence of $\rho_n(e)$

$$\sum_{n\in\mathbb{N}} \frac{1}{b_1(e)} \big(a_n(e) - b_n(e)\big)^2 \leq \frac{1}{\rho(e)}.$$

Rearranging:

$$\sum_{n\in\mathbb{N}} \big(a_n(e) - b_n(e)\big)^2 \leq \frac{b_1(e)}{\rho(e)} = \frac{\int \mathcal{V}(x)\big(1 - u(x,e)\big)\ \mathrm{dx}}{\int K_e \mathcal{V}(x)\ \mathrm{dx}} \leq \frac{\int \mathcal{V}(x)\ \mathrm{dx}}{\int K_e \mathcal{V}(x)\ \mathrm{dx}}.$$

The right-hand side is a continuous function, so on every compact interval $[e_1, e_2]$ it takes on a maximum, we denote as C:

$$C \geq \sum_{n\in\mathbb{N}} \big(a_n(e) - b_n(e)\big)^2.$$

Uniform convergence of $\rho_n(e)$

$$
\begin{aligned}
C &\geq \sum_{n\in\mathbb{N}} \big(a_n(e) - b_n(e)\big)^2 \\
&\geq \sum_{n\leq N} \big(a_n(e) - b_n(e)\big)^2 \\
&\geq N\big(a_N(e) - b_N(e)\big)^2
\end{aligned}
$$

Uniform convergence of $\rho_n(e)$

$$
\begin{aligned}
C &\geq \sum_{n\in\mathbb{N}} \left(a_n(e) - b_n(e)\right)^2 \\
&\geq \sum_{n\leq N} \left(a_n(e) - b_n(e)\right)^2 \\
&\geq N\left(a_N(e) - b_N(e)\right)^2
\end{aligned}
$$

We get:

$$
\left(a_N(e) - b_N(e)\right)^2 \leq \frac{C}{N} \Rightarrow \frac{1}{\rho_n} \text{ converges uniformly} \Rightarrow \rho \text{ is continuous}
$$

Uniform convergence of $\rho_n(e)$

$$C \geq \sum_{n\in\mathbb{N}} \left(a_n(e) - b_n(e)\right)^2$$
$$\geq \sum_{n\leq N} \left(a_n(e) - b_n(e)\right)^2$$
$$\geq N\left(a_N(e) - b_N(e)\right)^2$$

We get:

$$\left(a_N(e) - b_N(e)\right)^2 \leq \frac{C}{N} \Rightarrow \frac{1}{\rho_n} \text{ converges uniformly} \Rightarrow \rho \text{ is continuous}$$

□

Limit behaviour of $\rho(e)$

Limit behaviour of $\rho(e)$

$e \to \infty$:

$$\rho(e) = \frac{2e}{\int \mathcal{V}(x)\big(1 - u(x,e)\big)\mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\mathrm{dx}} \overset{e\to\infty}{\to} \infty$$

Limit behaviour of $\rho(e)$

$e \to \infty$:

$$\rho(e) = \frac{2e}{\int \mathcal{V}(x)\big(1 - u(x,e)\big)\mathrm{d}x} \geq \frac{2e}{\int \mathcal{V}(x)\mathrm{d}x} \stackrel{e\to\infty}{\to} \infty$$

$e \to 0$:

$$\begin{aligned} \rho(e) &= \frac{1}{\int u(x,e)\ \mathrm{d}x} \\ &\leq \frac{1}{\int u_1(x,e)\ \mathrm{d}x} \\ &= \frac{1}{\int Y_{4e} * \big(\mathcal{V}(1-u_1)\big)\ \mathrm{d}x} \\ &= \frac{4e}{\int \big(\mathcal{V}(1-u_1)\big)\ \mathrm{d}x} \stackrel{e\to 0}{\to} 0 \end{aligned}$$

Uniqueness of ρ and u

Uniqueness

Let $\tilde{u}$ be another non-negative integrable solution to equation (1), with

$$\tilde{\rho} = \frac{2e}{\int \left(1 - \tilde{u}(x)\right)\mathcal{V}(x)\ \mathrm{dx}}.$$

Uniqueness

Let $\tilde{u}$ be another non-negative integrable solution to equation (1), with

$$\tilde{\rho} = \frac{2e}{\int \big(1 - \tilde{u}(x)\big)\mathcal{V}(x)\ \mathrm{dx}}.$$

We first prove $\tilde{u} \geq u_n$ for all $n \in \mathbb{N}_0$ by induction:

Uniqueness

Let $\tilde{u}$ be another non-negative integrable solution to equation (1), with

$$\tilde{\rho} = \frac{2e}{\int \big(1 - \tilde{u}(x)\big)\mathcal{V}(x)\ \mathrm{dx}}.$$

We first prove $\tilde{u} \geq u_n$ for all $n \in \mathbb{N}_0$ by induction:

$n = 0$: $\tilde{u} \geq 0 = u_0$

Uniqueness

Let $\tilde{u}$ be another non-negative integrable solution to equation (1), with

$$\tilde{\rho} = \frac{2e}{\int \big(1 - \tilde{u}(x)\big)\mathcal{V}(x)\ \mathrm{dx}}.$$

We first prove $\tilde{u} \geq u_n$ for all $n \in \mathbb{N}_0$ by induction:

$n = 0$: $\tilde{u} \geq 0 = u_0$

$n \in \mathbb{N}$:

$$\tilde{\rho} = \frac{2e}{\int \mathcal{V}(x)\big(1 - \tilde{u}(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\big(1 - u_{n-1}(x)\big)\ \mathrm{dx}} = \rho_{n-1}.$$

Uniqueness

Let $\tilde{u}$ be another non-negative integrable solution to equation (1), with

$$\tilde{\rho} = \frac{2e}{\int \big(1 - \tilde{u}(x)\big)\mathcal{V}(x)\ \mathrm{dx}}.$$

We first prove $\tilde{u} \geq u_n$ for all $n \in \mathbb{N}_0$ by induction:

$n = 0$: $\tilde{u} \geq 0 = u_0$

$n \in \mathbb{N}$:

$$\tilde{\rho} = \frac{2e}{\int \mathcal{V}(x)\big(1 - \tilde{u}(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\big(1 - u_{n-1}(x)\big)\ \mathrm{dx}} = \rho_{n-1}.$$

This implies

$$\tilde{\rho}(\tilde{u} * \tilde{u})(x) \geq \rho_{n-1}\big(u_{n-1} * u_{n-1}\big)(x).$$

Uniqueness

Let $\tilde{u}$ be another non-negative integrable solution to equation (1), with

$$\tilde{\rho} = \frac{2e}{\int \big(1 - \tilde{u}(x)\big)\mathcal{V}(x)\ \mathrm{dx}}.$$

We first prove $\tilde{u} \geq u_n$ for all $n \in \mathbb{N}_0$ by induction:

$n = 0$: $\tilde{u} \geq 0 = u_0$

$n \in \mathbb{N}$:

$$\tilde{\rho} = \frac{2e}{\int \mathcal{V}(x)\big(1 - \tilde{u}(x)\big)\ \mathrm{dx}} \geq \frac{2e}{\int \mathcal{V}(x)\big(1 - u_{n-1}(x)\big)\ \mathrm{dx}} = \rho_{n-1}.$$

This implies

$$\tilde{\rho}(\tilde{u} * \tilde{u})(x) \geq \rho_{n-1}(u_{n-1} * u_{n-1})(x).$$

This implies

$$\tilde{u}(x) - u_n(x) = 2eK_e\big(\tilde{\rho}(\tilde{u} * \tilde{u}) - \rho_{n-1}(u_{n-1} * u_{n-1})\big)(x) \geq 0 \Rightarrow \tilde{u} \geq u_n.$$

Uniqueness

In the limit, we get

$$\tilde{u} \geq u \qquad \text{and} \qquad \tilde{\rho} \geq \rho.$$

Uniqueness

In the limit, we get

$$\tilde{u} \geq u \qquad \text{and} \qquad \tilde{\rho} \geq \rho.$$

But we also get

$$\int \tilde{u}(x)\ \mathrm{dx} = \frac{1}{\tilde{\rho}} \leq \frac{1}{\rho} = \int u(x)\ \mathrm{dx}.$$

Uniqueness

In the limit, we get

$$\tilde{u} \geq u \qquad \text{and} \qquad \tilde{\rho} \geq \rho.$$

But we also get

$$\int \tilde{u}(x)\ \mathrm{dx} = \frac{1}{\tilde{\rho}} \leq \frac{1}{\rho} = \int u(x)\ \mathrm{dx}.$$

Since $\tilde{u} \geq u$ but $\int \tilde{u}\mathrm{dx} \leq \int u\mathrm{dx}$, we must have

$$u(x) = \tilde{u}(x) \text{ for almost all } x \in \mathbb{R}^d).$$

But since $u, \tilde{u}$ are continuous, we must have

$$u(x) = \tilde{u}(x) \text{ for all } x \in \mathbb{R}^d).$$

Uniqueness

In the limit, we get

$$\tilde{u} \geq u \qquad \text{and} \qquad \tilde{\rho} \geq \rho.$$

But we also get

$$\int \tilde{u}(x)\ \mathrm{dx} = \frac{1}{\tilde{\rho}} \leq \frac{1}{\rho} = \int u(x)\ \mathrm{dx}.$$

Since $\tilde{u} \geq u$ but $\int \tilde{u}\mathrm{dx} \leq \int u\mathrm{dx}$, we must have

$$u(x) = \tilde{u}(x) \text{ for almost all } x \in \mathbb{R}^d).$$

But since $u, \tilde{u}$ are continuous, we must have

$$u(x) = \tilde{u}(x) \text{ for all } x \in \mathbb{R}^d).$$