Reading seminar summer 2021

An effective theory for interacting Bose gases

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Theorem 1.5 in [2]

Our system

$$\boxed{(-\Delta + 4e + \mathcal{V}(x))u(x) = \mathcal{V}(x) + 2e\rho(u * u)(x)\ ,} \tag{1.1}$$

$$e = \frac{\rho}{2}\int (1 - u(x))\mathcal{V}(x)\ dx\ . \tag{1.2}$$

The goal of today

Theorem 1.5 (decay of u at infinity) In all dimensions, provided $\mathcal{V}$ is spherically symmetric with $\int |x|^2 \mathcal{V} dx < \infty$ in addition to satisfying the hypotheses imposed in Theorem 1.3, all integrable solutions of (1.1)-(1.2) with $u(x) \leqslant 1$ for all x satisfy

$$\int |x| u(x) dx = \infty \quad \text{and} \quad \int |x|^r u(x) dx < \infty \quad \text{for all } 0 < r < 1 \ . \tag{1.25}$$

Thus, if $u(x) \sim |x|^{-m}$ for some m, the only possibility is $m = d + 1$. Under stronger assumptions on the potential, this is actually the case. For $d = 3$, if $\mathcal{V}$ is non-negative, square-integrable, spherically symmetric (that is, $\mathcal{V}(x) = \mathcal{V}(|x|)$), and, for $|x| > R$,

$$\mathcal{V}(|x|) \leqslant A e^{-B|x|} \tag{1.26}$$

for some $A, B > 0$ then there exists $\alpha > 0$ such that

$$u(x) \underset{|x| \to \infty}{\sim} \frac{\alpha}{|x|^4}. \tag{1.27}$$

Theorem 1.1(Positivity) Suppose that $\mathcal{V}$ is non-negative and integrable and that u is an integrable solution of (1.1)-(1.2) such that $u(x) \leqslant 1$ for all x. Then $u(x) \geqslant 0$ for all x, and all such solutions have fairly slow decay at infinity in that they satisfy

$$\int |x|u(x)dx = \infty \ . \tag{1.5}$$

Thus, any physical solutions of (1.1)-(1.2) must necessarily satisfy the *pair* of inequalities

$$0 \leqslant u(x) \leqslant 1 \quad \text{for all } x \ . \tag{1.6}$$

Theorem 1.3 (existence and uniqueness) Let $\mathcal{V} \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > \max\{\frac{d}{2}, 1\}$, be non-negative. Then there is a constructively defined continuous function $\rho(e)$ on $(0, \infty)$ such that $\lim_{e\to 0} \rho(e) = 0$ and $\lim_{e\to\infty} \rho(e) = \infty$ and such that for any $e \geqslant 0$ and $\rho = \rho(e)$, the system (1.1) and (1.2) has a unique integrable solution $u(x)$ satisfying $u(x) \leqslant 1$. Moreover, if $\rho \neq \rho(e)$, the system (1.1) and (1.2) has *no* integrable solution $u(x)$ satisfying (1.6).

Remark

- We do not assume here that the potential is radially symmetric. However, the uniqueness statement implies that u is radially symmetric whenever $\mathcal{V}$ is radially symmetric.
- The function $\rho(e)$ is the *density function*, which specifies the density as a function of the energy. Thus, our system together with (1.6) constrains the parameters e and ρ to be related by a strict functional relation $\rho = \rho(e)$. In most of the early literature on the Bose gas, ρ is taken as the independent parameter, as suggested by (**??**): One puts N particles in a box of volume N/ρ, and seeks to find the ground state energy per particle, e, as a function of ρ. Our theorem goes in the other direction, with ρ specified as a function of e. We prove that $e \mapsto \rho(e)$ is continuous, and we conjecture that $\rho(e)$ is a strictly monotone increasing function. In that case, the functional relation could be inverted, and we would have a well-defined function $e(\rho)$.
- Since $\lim_{e\to 0} \rho(e) = 0$ and $\lim_{e\to\infty} \rho(e) = \infty$, the continuity of $e \to \rho(e)$ implies that for each $\rho \in (0,\infty)$ there is *at least one* e such that $\rho(e) = \rho$.

Proof of Theorem 1.5 I

- The first part of (1.25) has already been proved in Theorem 1.1 without the additional assumption on the potential.
- For the second part of (1.25), by the first remark after Theorem 1.3, u is also radial, and hence $\mathcal{V}(1-u)$ is non-negative and radial. It then follows from the hypotheses on $\mathcal{V}$ that $g := 2\rho e Y_{4e} * Y_{4e} * [\mathcal{V}(1-u)]$ satisfies

$$\int |x|^2 g(x)dx < \infty \quad \text{and} \quad \int xg(x)dx = 0 \ .$$

Then, as explained in Section 2, if $f := 2e\rho Y_{4e} * u$, $f - f * f = g \geqslant 0$, and then by [CJLL20, Theorem 4], the second part of (1.25) follows.

Proof of Theorem 1.5 II

Note that if

$$u(|x|) \underset{|x|\to\infty}{\sim} \frac{\alpha}{|x|^m}$$

for some $\alpha > 0$, then the only choice of m that is consistent with (1.25) is $m = d + 1$.

It can be seen by the following:

$$\int_{|x|>R} |x|^r \frac{1}{|x|^m} \mathrm{d}x \sim \int_R^\infty \rho^{r-m+d-1} \mathrm{d}\rho < \infty \iff r - m + d - 1 < -1.$$

Then, we have $r < 1$ only when $m = d + 1$.

Proof of Theorem 1.5 III

We now specialize to $d = 3$, with the additional assumption on $\mathcal{V}$.
Fourier transform of u:

$$\hat{u}(|k|) = \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e} + 1\right)^2 - S(|k|)}\right)$$

where S is defined by

$$S(|k|) := \frac{\rho}{2e}\int e^{ikx}(1 - u(|x|))\mathcal{V}(|x|)\ dx.$$

We split

$$\hat{u}(|k|) = \widehat{\mathcal{U}}_1(|k|) + \widehat{\mathcal{U}}_2(|k|) \tag{5.5}$$

with

$$\widehat{\mathcal{U}}_1(|k|) := \frac{2eS(|k|)}{\rho(4e + k^2)}$$

Proof of Theorem 1.5 IV

so that, taking the large $|k|$ limit in (4.25),

$$\widehat{\mathcal{U}}_2(|k|) = O(|k|^{-6} S^2(|k|)) \tag{5.7}$$

so $\widehat{\mathcal{U}}_2$ is integrable.

Proof of Theorem 1.5 - Decay of $\mathcal{U}_1$ I

We first show that

$$\mathcal{U}_1(|x|) := \frac{1}{(2\pi)^3} \int e^{-ikx} \widehat{\mathcal{U}}_1(|k|) \; dk$$

decays exponentially in $|x|$. We have

$$\mathcal{U}_1(|x|) = (-\Delta + 1)^{-1}(1 - u(|x|))\mathcal{V}(|x|) = Y_1 * ((1-u)\mathcal{V})(|x|)$$

with

$$Y_1(|x|) := \frac{e^{-|x|}}{4\pi|x|}.$$

Therefore, by (1.26),

$$\mathcal{U}_1(|x|) \leqslant \frac{A}{4\pi} \int_{|y|>R} \frac{e^{-|x-y|-B|y|}}{|x-y|} \; dy + \frac{1}{4\pi} \int_{|y|<R} \frac{e^{-|x-y|}}{|x-y|} \mathcal{V}(|y|) \; dy$$

Proof of Theorem 1.5 - Decay of $\mathcal{U}_1$ II

so, denoting $b := \min(B, 1)$,

$$\mathcal{U}_1(|x|) \leqslant \frac{A}{4\pi} \int \frac{e^{-b(|x-y|+|y|)}}{|x-y|}\, dy + \frac{e^{-(|x|-R)}}{4\pi(|x|-R)} \int \mathcal{V}(|y|)\, dy$$

and since

$$\frac{A}{4\pi} \int \frac{e^{-b(|x-y|+|y|)}}{|x-y|}\, dy = \frac{A}{4\pi} \int \frac{e^{-b(|y|+|y+x|)}}{|y|}\, dy$$
$$\leq \frac{A}{4\pi} \int_{y\leq|x|} \frac{e^{-b|x|}}{|y|}\, dy + \frac{A}{4\pi} \int_{y>|x|} \frac{e^{-b|y|}}{|y|}\, dy \leq C(b) e^{-b|x|}(|x|^2 + |x| + 1)$$

we have

$$\mathcal{U}_1(|x|) \leqslant C(b) e^{-b|x|}(|x|^2 + |x| + 1) + \frac{e^{-(|x|-R)}}{4\pi(|x|-R)} \int \mathcal{V}(|y|)\, dy. \quad (5.14)$$

Proof of Theorem 1.5 - Analyticity of $\mathcal{U}_2$

We now turn to

$$\mathcal{U}_2(|x|) := \frac{1}{(2\pi)^3} \int e^{-ikx} \widehat{\mathcal{U}}_2(|k|)\, dk = \frac{1}{4i\pi^2|x|} \sum_{\eta=\pm} \eta \int_0^\infty e^{i\eta\kappa|x|} \kappa \widehat{\mathcal{U}}_2(\kappa)\, d\kappa. \tag{5.15}$$

We start by proving some analytic properties of $\widehat{\mathcal{U}}_2$, which, we recall from (4.25) and (5.5), is

$$\widehat{\mathcal{U}}_2(|k|) = \frac{1}{\rho} \left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e} + 1\right)^2 - S(|k|) - \frac{2eS(|k|)}{4e + k^2}} \right).$$

Proof of Theorem 1.5 - 2-1

First of all, S is analytic in a strip about the real axis:

$$S(\kappa) = 4\pi \int_0^\infty \mathrm{sinc}(\kappa r) r^2 \mathcal{V}(r)(1-u(r))\, dr, \quad \mathrm{sinc}(\xi) := \frac{\sin(\xi)}{\xi}$$

so

$$\partial^n S(\kappa) = 4\pi \int_0^\infty \partial^n \mathrm{sinc}(\kappa r) r^{n+2} \mathcal{V}(r)(1-u(r))\, dr.$$

We will show that if $\mathcal{I}m(\kappa) \leqslant \frac{B}{2}$, then there exists $C > 0$ which only depends on A and B such that

$$|\partial^n S(\kappa)| \leqslant n! C^n. \tag{5.19}$$

Because the Taylor series of S at κ converges, S is analytic in a strip. In particular, if we define the strip

$$H_\tau := \{z :\ |\mathcal{I}m(z)| \leqslant r^{-\tau},\ \mathcal{R}e(z) > 0\} \quad \text{and} \quad r > \left(\frac{B}{2}\right)^{-\frac{1}{\tau}}$$

with $0 < \tau < 1$. Then S is analytic in H_τ.

Proof of Theorem 1.5 - 2-1-1 I

We first treat the case $|\kappa| \leqslant \frac{B}{2}$. We have

$$\mathrm{sinc}(\xi) = \sum_{p=0}^{\infty} \frac{(-1)^p \xi^{2p}}{(2p+1)!}$$

so

$$\partial^n \mathrm{sinc}(\xi) = \sum_{p=\lceil \frac{n}{2} \rceil}^{\infty} \frac{(-1)^p \xi^{2p-n}}{(2p+1)(2p-n)!}.$$

Therefore

$$|\partial^n \mathrm{sinc}(\xi)| \leqslant \sum_{p=\lceil \frac{n}{2} \rceil}^{\infty} \frac{|\xi|^{2p-n}}{(2p-n)!} \leqslant \cosh(|\xi|).$$

Thus,

$$|\partial^n S(\kappa)| \leqslant 4\pi \int_0^{\infty} \cosh(|\kappa| r) r^{n+2} \mathcal{V}(r)(1-u(r))\, dr$$

Proof of Theorem 1.5 - 2-1-1 II

so, by (1.26),

$$|\partial^n S(\kappa)| \leqslant 4A\pi \int_R^\infty \cosh(|\kappa| r) r^{n+2} e^{-Br}\, dr + 4\pi \int_0^R \cosh(|\kappa| r) r^{n+2} \mathcal{V}(r)\, dr$$

and

$$|\partial^n S(\kappa)| \leqslant 8A\pi \int_0^\infty r^{n+2} e^{-(B-|\kappa|)r}\, dr + 8\pi e^{|\kappa| R} R^n \int r^2 \mathcal{V}(r)\, dr$$

which, if $|\kappa| \leqslant \frac{B}{2}$, implies that

$$8A\pi \int_0^\infty r^{n+2} e^{-(B-|\kappa|)r}\, dr \leqslant 8A\pi \int_0^\infty r^{n+2} e^{-\frac{B}{2} r}\, dr = \frac{2^{n+6} A\pi}{B^{n+3}} (n+2)!$$

and

$$8\pi e^{|\kappa| R} R^{n+2} \int \mathcal{V}(r)\, dr \leqslant 8\pi e^{\frac{B}{2} R} R^n \int r^2 \mathcal{V}(r)\, dr.$$

Proof of Theorem 1.5 - 2-1-2 I

We now turn to $|\kappa| \geqslant \frac{B}{2}$:

$$\partial^n \mathrm{sinc}(\xi) = \sum_{p=0}^{n} \binom{n}{p} \partial^p \sin(\xi) \frac{(n-p)!(-1)^{n-p}}{\xi^{n-p+1}}$$

so

$$|\partial^n \mathrm{sinc}(\xi)| \leqslant 2e^{\mathcal{I}m(\xi)} \sum_{p=0}^{n} \frac{n!}{p!} |\xi|^{-(n-p+1)}.$$

Therefore,

$$|\partial^n S(\kappa)| \leqslant 8\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} \int_0^\infty e^{\mathcal{I}m(\kappa)r} r^{p+1} \mathcal{V}(r)(1-u(r))\, dr$$

so, by (1.26),

$$|\partial^n S(\kappa)| \leqslant \sigma_1 + \sigma_2$$

Proof of Theorem 1.5 - 2-1-2 II

with

$$\sigma_1 := 8A\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} \int_R^\infty r^{p+1} e^{-(B-\mathcal{I}m(\kappa))r}\ dr$$

and

$$\sigma_2 := 8\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} \int_0^R r^{p+1} e^{\mathcal{I}m(\kappa)r} \mathcal{V}(r)\ dr.$$

Furthermore,

$$\sigma_1 = 8A\pi n! \sum_{p=0}^{n} \frac{p+1}{(B-\mathcal{I}m(\kappa))^{p+2}|\kappa|^{n-p+1}}$$

so, as long as $|\kappa| \geqslant \frac{1}{2}B$ and $\mathcal{I}m(\kappa) \leqslant \frac{1}{2}B$,

$$\sigma_1 \leqslant \frac{2^{n+6}A\pi}{B^{n+3}} n! \sum_{p=0}^{n}(p+1) = \frac{2^{n+5}A\pi}{B^{n+3}}(n+2)!.$$

Proof of Theorem 1.5 - 2-1-2 III

In addition,

$$\sigma_2 \leqslant 8\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} R^{p-1} e^{\mathcal{I}m(\kappa)R} \int_0^R r^2 \mathcal{V}(r)\ dr$$

so

$$\begin{aligned}\sigma_2 &\leqslant 8\pi \sum_{p=0}^{n} \frac{n! 2^{n-p+1}}{p! B^{n-p+1}} R^{p-1} e^{\mathcal{I}m(\kappa)R} \int_0^R r^2 \mathcal{V}(r)\ dr \\ &\leqslant \frac{2^{n+4}\pi}{RB^{n+1}} n! e^{RB} \int_0^R r^2 \mathcal{V}(r)\ dr.\end{aligned}$$

Proof of Theorem 1.5 - 2-2

We have thus proved that S is analytic in H_τ, which implies that the singularities of $\widehat{\mathcal{U}}_2$ in H_τ all come from the branch points of $\sqrt{F(|k|)}$ with $F(|k|) := (\frac{k^2}{4e}+1)^2 - S(|k|)$. For $\kappa \in \mathbb{R}$,

$$|S(\kappa)| \leqslant 1$$

so, for $\kappa \in \mathbb{R}$,

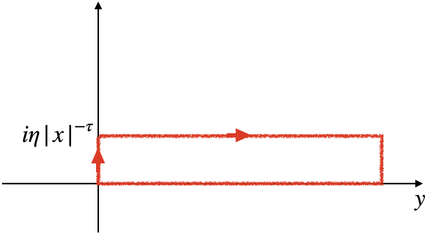
$$F(\kappa) \geqslant \frac{\kappa^2}{2e}.$$

Therefore, since F is analytic in a strip around the real axis, there exists an open set containing the real axis in which F has one and only one root, at 0. Thus the only branch point of $\sqrt{F}$ on the real axis is 0. Thus, $\widehat{\mathcal{U}}_2$ is analytic in H_τ.

Decay of $\mathcal{U}_2$ I

We deform the integral to the path

$$\{i\eta y,\ 0 < y < |x|^{-\tau}\} \cup \{i\eta|x|^{-\tau} + y,\ y > 0\}$$



Decay of $\mathcal{U}_2$ II

and find

$$\int_0^\infty e^{i\eta\kappa|x|}\kappa\widehat{\mathcal{U}}_2(\kappa)\ d\kappa = I_1 + I_2 \tag{5.43}$$

with

$$I_1 := -\int_0^{|x|^{-\tau}} e^{-y|x|}y\widehat{\mathcal{U}}_2(i\eta y)\ dy$$

and

$$I_2 := e^{-|x|^{1-\tau}}\int_0^\infty e^{i\eta y|x|}(i\eta|x|^{-\tau}+y)\widehat{\mathcal{U}}_2(i\eta|x|^{-\tau}+y)\ dy.$$

3-1 I

We first estimate I_1. We expand S: For $\beta > 0$,since S is analytic and symmetric, and $|S(|k|)| \leqslant 1$,

$$S(\kappa) = 1 - \beta\kappa^2 + O(|\kappa|^4).$$

Therefore, $y \mapsto \widehat{\mathcal{U}}_2(iy)$ is $\mathcal{C}^2$ for $y \neq 0$, and

$$\begin{aligned}
&\widehat{\mathcal{U}}_2(|k|) \\
&= \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(|k|)} - \frac{2eS(|k|)}{4e + k^2}\right) \\
&= \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\frac{k^4}{16e^2} + \frac{k^2}{2e} + 1 - 1 + \beta k^2 + O(|k|^4)} - \frac{2e}{4e+k^2}(1 + \beta k^2 + O(|k|^4))\right) \\
&= \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\frac{k^2}{2e} + \beta k^2 + O(|k|^4)} - \frac{2e}{4e+k^2}(1 - \beta k^2 + O(|k|^4))\right) \\
&= \frac{1}{\rho}\left(1 - k\sqrt{\frac{1}{2e} + \beta} - \frac{1}{2} + O(k^2)\right)
\end{aligned}$$

3-1 II

Thus,

$$\widehat{\mathcal{U}}_2(i\eta y) = \frac{1}{2\rho} - \frac{i\eta y}{\rho}\sqrt{\frac{1}{2e} + \beta} + O(y^2)$$

Furthermore,

$$-\int_0^{|x|^{-\tau}} e^{-y|x|} y \; dy = -\frac{1}{|x|^2} + \frac{1 + |x|^{1-\tau}}{|x|^2} e^{-|x|^{1-\tau}}$$

$$-\int_0^{|x|^{-\tau}} e^{-y|x|} y^2 \; dy = -\frac{2}{|x|^3} + \frac{1 + |x|^{1-\tau}(2 + x^{1-\tau})}{|x|^3} e^{-|x|^{1-\tau}}$$

and

$$-\int_0^{|x|^{-\tau}} e^{-y|x|} y^3 \; dy = O(|x|^{-4})$$

3-1 III

$$I_1 = -\frac{1}{2\rho|x|^2} + \frac{2i\eta}{\rho|x|^3}\sqrt{\frac{1}{2e}+\beta} + O(|x|^{-4})$$

so

$$\frac{1}{4i\pi^2|x|}\sum_{\eta=\pm}\eta I_1 = \frac{1}{\pi^2\rho|x|^4}\sqrt{\frac{1}{2e}+\beta} + O(|x|^{-5}). \qquad (5.52)$$

3-2

We now bound I_2. Recall that, for $\kappa \in \mathbb{R}$, $|S(\kappa)| \leqslant 1$. Recalling (5.19),

$$|S(\kappa + i\eta|x|^{-\tau})| \leqslant \sum_{n=0}^{\infty} \frac{1}{n!} |\partial^n S(\kappa)|^n |x|^{-n\tau} \leqslant \frac{1}{1 - C|x|^{-\tau}} \leqslant 2$$

provided $|x|^{\tau} > 2C$. Therefore, for large κ, by (5.7),

$$|\widehat{\mathcal{U}}_2(\kappa + i\eta)| = O(\kappa^{-4})$$

so

$$I_2 \leqslant C' e^{-|x|^{1-\tau}} \tag{5.55}$$

for some constant $C' > 0$.

3-3

Inserting (5.52) and (5.55) into (5.43) and (5.15), we find that

$$\mathcal{U}_2(|x|) = \frac{1}{\pi^2 \rho |x|^4} \sqrt{\frac{1}{2e} + \beta} + O(|x|^{-5})$$

which, using (2.10), concludes the proof of the theorem. □

Theorem 1.2 in [4]

Theorem 1.2 (Large $|x|$ asymptotics of u) If $(1+|x|^4)v(x) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, then

$$\rho u(x) = \frac{\sqrt{2+\beta}}{2\pi^2\sqrt{e}} \frac{1}{|x|^4} + R(x)$$

where

$$\beta = \rho \int |x|^2 v(1-u)dx \leqslant \rho \|x^2 v\|_1,$$

and where $|x|^4 R(x)$ is in $L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, uniformly in e on all compact sets. Moreover, for every $\rho_0 > 0$, there is a constant C that only depends on ρ_0 such that for all x, for all $\rho < \rho_0$,

$$u(x) \leqslant \min\left\{1, \frac{C}{\rho e^{\frac{1}{2}}|x|^4}\right\} \ .$$

Proof of Theorem 1.2: Preparation I

Let

$$\kappa := \frac{|k|}{2\sqrt{e}}. \tag{2.1}$$

Then we have

$$\rho\widehat{u} = (\kappa^2 + 1)\left(1 - \sqrt{1 - \frac{\frac{\rho}{2e}\widehat{S}}{(\kappa^2+1)^2}}\right). \tag{2.2}$$

For small κ, since $x^4 v$ is integrable, $\widehat{S}$ is $\mathcal{C}^4$

$$\frac{\rho}{2e}\widehat{S} = 1 - \beta\kappa^2 + O(e^2\kappa^4) \tag{2.3}$$

and β is defined in (29):

$$\beta = -\frac{\rho}{4e}\partial_\kappa^2\widehat{S} \leqslant \rho\|x^2 v\|_1. \tag{2.4}$$

Proof of Theorem 1.2: Preparation II

Therefore, defining

$$\widehat{U}_1 := (\kappa^2+1)^{-2}\left(1-\sqrt{1-\frac{(1-\beta\kappa^2)}{(\kappa^2+1)^2}}\right)$$

$\widehat{U}_1$ coincides with $\widehat{u}$ asymptotically as $\kappa \to 0$ and we chose the prefactor $(\kappa^2+1)^{-2}$ in such a way that $\widehat{U}_1$ is integrable. Define the remainder term

$$\widehat{U}_2 := \rho\widehat{u} - \widehat{U}_1 = (\kappa^2+1)\left(1-\sqrt{1-2\zeta_1}\right) - (\kappa^2+1)^{-2}\left(1-\sqrt{1-2\zeta_2}\right)$$

with

$$\zeta_1 := \frac{\frac{\rho}{4e}\widehat{S}}{(\kappa^2+1)^2}, \quad \zeta_2 := \frac{1-\beta\kappa^2}{2(\kappa^2+1)^2}. \tag{2.7}$$

$U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx} \widehat{U}_1(k)$ I

We write

$$\sqrt{1 - \frac{1-\beta\kappa^2}{(1+\kappa^2)^2}} = \frac{\kappa}{1+\kappa^2}\sqrt{2+\beta+\kappa^2} = \frac{1}{\pi}\frac{|\kappa|(2+\beta+\kappa^2)}{1+\kappa^2}\int_0^\infty \frac{1}{2+\beta+t+\kappa^2} t^{-1/2} dt \ .$$

Therefore,

$$\widehat{U}_1 := (\kappa^2+1)^{-2} - \frac{\kappa}{\pi}(\kappa^2+1)^{-2}\left(1+(\beta+1)\frac{1}{1+\kappa^2}\right)\int_0^\infty \frac{1}{2+\beta+t+\kappa^2} t^{-1/2} dt.$$

We take the inverse Fourier transform of $\widehat{U}_1$, recalling the definition of κ (2.1)

$$U_1(x) = \frac{e^{\frac{3}{2}}}{\pi} e^{-2\sqrt{e}|x|} - \frac{1}{\pi}\left(\delta(x) + \frac{(\beta+1)e}{\pi}\frac{e^{-2\sqrt{e}|x|}}{|x|}\right) * f_1 * f_2 \qquad (2.10)$$

where

$$f_1(x) := \frac{e^{\frac{3}{2}}}{\pi^3}\int dk \ e^{-ik(2\sqrt{e}x)}\frac{|k|}{(k^2+1)^2}$$

$U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx} \widehat{U}_1(k)$ II

and

$$f_2(x) := \frac{e^{\frac{3}{2}}}{\pi^3} \int dk\, e^{-ik(2\sqrt{e}x)} \int_0^\infty \frac{dt}{\sqrt{t}} \frac{1}{2+\beta+t+k^2} = \frac{e}{\pi|x|} \int_0^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)} t^{-1/2} dt\,,$$

now, for all $T > 0$,

$$\begin{aligned}
&\int_0^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)} t^{-1/2}\, dt \\
&= \int_0^T e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)} t^{-1/2} dt + \int_T^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)} t^{-1/2}\, dt \\
&\leqslant \int_0^T e^{-\sqrt{2+\beta}} e^{-\sqrt{t}}(2\sqrt{e}|x|) t^{-1/2} dt + \int_T^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)} t^{-1/2}\, dt \\
&= 2(1-e^{-\sqrt{T}}) e^{-\sqrt{2+\beta}(2\sqrt{e}|x|)} + \frac{1}{\sqrt{e}|x|} e^{-\sqrt{T}(2\sqrt{e}|x|)} \\
&\leqslant 2T^{1/2} e^{-\sqrt{2+\beta}(2\sqrt{e}|x|)} + \frac{1}{\sqrt{e}|x|} e^{-\sqrt{T}(2\sqrt{e}|x|)}. \qquad (2.13)
\end{aligned}$$

$U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx} \widehat{U}_1(k)$ III

Where we have use that

$$\exp(-x) = 1 - x + \frac{1}{2}x^2 + O(x^3)$$

for the last inequality.
Choosing $T = 2 + \beta$, we see that for large $(2\sqrt{e}|x|)$, $0 \leqslant f_2(x) \leqslant Ce^{-\sqrt{2+\beta}(2\sqrt{e}|x|)}$. Furthermore,

$$f_1(x) = \frac{e^{\frac{3}{2}}}{\pi^3} \int dk \ e^{-ik(2\sqrt{e}x)} \frac{1}{|k|} \frac{k^2}{(k^2+1)^2} = \frac{e^{\frac{3}{2}}}{\pi^3} \frac{1}{|x|^2} * g, \quad g(x) = \frac{(1-\sqrt{e}|x|)e^{-(2\sqrt{e})|x|}}{|x|}$$

Using

$$\frac{1}{|x-y|^2} = \frac{1}{|x|^2} + \frac{-|y|^2 + 2x \cdot y}{|x|^2|x-y|^2}$$

$U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx} \widehat{U}_1(k)$ IV

twice and the fact that $g(y)$ is even, integrates to zero, and $\int yg(y)\, dy = 0$,

$$f_1(x) = \frac{1}{|x|^4} \frac{e^{\frac{3}{2}}}{\pi^3} \left(- \int_{\mathbb{R}^3} |y|^2 g(y) \mathrm{d}y + \int_{\mathbb{R}^3} \frac{(-|y|^2 + 2x \cdot y)^2}{|x-y|^2} g(y) \mathrm{d}y \right) \tag{2.16}$$

We compute $\int_{\mathbb{R}^3} |y|^2 g(y) \mathrm{d}y = -\frac{3\pi}{2e^2}$, and then using the symmetry of g once more,

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^3} \frac{(x \cdot y)^2}{|x-y|^2} g(y) \mathrm{d}y = \frac{1}{3} \int_{\mathbb{R}^3} |y|^2 g(y) \mathrm{d}y = -\frac{\pi}{2e^2} \ ,$$

Therefore,

$$\lim_{|x|\to\infty} |x|^4 f_1(x) = -\frac{1}{2\pi^2 \sqrt{e}} \qquad \text{and} \qquad \lim_{|x|\to\infty} |x|^4 U_1(x) = \frac{1}{2\pi^2 \sqrt{e}} \sqrt{2+\beta}. \tag{2.18}$$

$U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx} \widehat{U}_1(k)$ V

We now turn to an upper bound of U_1. First of all, if $|x| \leqslant \frac{1}{\sqrt{e}}$, then by (32) and (2.16),

$$f_1(x) \geqslant 0$$

and if $|x| > \frac{1}{\sqrt{e}}$, then

$$f_1(x) \geqslant -\frac{1}{|x|^4} \frac{e^2}{\pi^3} \int_{\mathbb{R}^3} \frac{(-|y|^2 + 2x \cdot y)^2}{|x-y|^2} e^{-(2\sqrt{e})|y|} \mathrm{d}y.$$

We split the integral into two parts: $|y-x| > |x|$ and $|y-x| < |x|$. We have, (recalling $|x| > \frac{1}{\sqrt{e}}$),

$$\int_{|y-x|>|x|} \frac{(-|y|^2 + 2x \cdot y)^2}{|x-y|^2} e^{-(2\sqrt{e})|y|} \mathrm{d}y \leqslant e^{-\frac{5}{2}} C$$

$U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx} \widehat{U}_1(k)$ VI

for some constant C (we use a notation where the constant C may change from one line to the next). Now,

$$\int_{|y-x|<|x|} \frac{(-|y|^2 + 2x\cdot y)^2}{|x-y|^2} e^{-(2\sqrt{e})|y|} \mathrm{d}y \leqslant e^{-\sqrt{e}|x|} \int_{|y-x|<|x|} \frac{(|y|^2 + 2|x||y|)^2}{|x-y|^2} \mathrm{d}y \leqslant |x|^5 e^{-\sqrt{e}}$$

Therefore, for all x,

$$f_1(x) \geqslant -\frac{1}{|x|^4} C(e^{-\frac{1}{2}} + e^2 |x|^4 e^{-\sqrt{e}|x|}).$$

Finally, by use (2.13),

$$|x|^4 \left(\delta(x) + \frac{(\beta+1)e}{\pi} \frac{e^{-2\sqrt{e}|x|}}{|x|} \right) * f_1 * f_2(x) \geqslant -Ce^{-\frac{1}{2}}.$$

All in all, by (2.10), (since $|x|^4 e^{\frac{3}{2}} e^{-2\sqrt{e}|x|} < Ce^{-\frac{1}{2}}$)

$$|x|^4 U_1(x) \leqslant Ce^{-\frac{1}{2}}. \tag{2.25}$$

$\Delta^2 \widehat{U}_2$ is integrable and square-integrable.

We use the fact that

$$16e^2\Delta^2 \equiv \partial_\kappa^4 + \frac{4}{\kappa}\partial_\kappa^3. \tag{2.26}$$

We have, by the Leibniz rule,

$$\partial_\kappa^n \widehat{U}_2 = \sum_{i=0}^{n} \binom{n}{i} \left(\partial_\kappa^{n-i}(\kappa^2+1)\partial_\kappa^i(1-\sqrt{1-2\zeta_1}) - \partial_\kappa^{n-i}(\kappa^2+1)^{-2}\partial_\kappa^i(1-\sqrt{1-2\zeta_2})\right). \tag{2.27}$$

Furthermore,

$$\partial_\kappa^n(1-\sqrt{1-2\zeta_j}) = \sum_{p=1}^{n} \partial_{\zeta_j}^p(1-\sqrt{1-2\zeta_j}) \sum_{\substack{l_1,\cdots,l_p\in\{1,\cdots,n\}\\ l_1+\cdots+l_p=n}} c_{l_1,\cdots,l_p}^{(p,n)} \prod_{i=1}^{n} \partial_\kappa^{l_i}\zeta_j \tag{2.28}$$

for some family of constants $c_{l_1,\cdots,l_p}^{(p,n)}$ which can easily be computed explicitly, but this is not needed. Now, since $S \geqslant 0$, $\frac{\rho}{1e}|\widehat{S}| \leqslant 1$, so $|\zeta_1| \leqslant \frac{1}{2}$ and $\zeta_1 = \frac{1}{2}$ if and only if $\kappa = 0$. Therefore, $\widehat{U}_2$ is bounded when κ is away from 0, so it suffices to show that $\Delta^2\widehat{U}_2$ is integrable and square integrable at infinity and at 0.

2-1

We first consider the behavior at infinity, and assume that κ is sufficiently large. The fact that $\partial_\kappa^{n-i}(\kappa^2+1)^{-2}\partial_\kappa^i(1-\sqrt{1-2\zeta_2})$ is integrable and square integrable at infinity follows immediately from (2.7). To prove the corresponding claim for ζ_1, we use the fact that $|x|^4 v$ square integrable, which implies that $\widehat{S}$ is as well. Therefore, by (2.7) for $0 \leqslant n \leqslant 4$, $\kappa^2\partial_\kappa^n\zeta_1$ is integrable at infinity, and, therefore, square-integrable at infinity. Furthermore, by (2.7), $\zeta_1 < \frac{1}{2} - \varepsilon$ for large κ, and $\partial^n\zeta_1$ is bounded, so $\partial_\kappa^{n-i}(\kappa^2+1)\partial_\kappa^i(1-\sqrt{1-2\zeta_1})$ is integrable and square integrable.

2-2 I

As $\kappa \to 0$

$$\zeta_i = \frac{1}{2}(1 - (\beta + 2)\kappa^2) + O(\kappa^4)$$

and since $\beta \geqslant 0$,

$$1 - 2\zeta_i \geqslant \kappa^2 + O(\kappa^4).$$

therefore, for $p \geqslant 1$

$$\partial^p_{\zeta_j}(1 - \sqrt{1 - 2\zeta_j}) = O(\kappa^{1-2p})$$

and, since ζ_i is $\mathcal{C}^4$, for $3 \leqslant n \leqslant 4$,

$$\partial\zeta_i = -(\beta + 2)\kappa + O(\kappa^3), \quad \partial^2\zeta_i = -(\beta + 2) + O(\kappa^2), \quad \partial^n\zeta_i = O(\kappa^{4-n}).$$

Therefore, for $1 \leqslant i \leqslant 4$, by (2.28)

$$\partial^i_\kappa(1 - \sqrt{1 - 2\zeta_1}) - \partial^i_\kappa(1 - \sqrt{1 - 2\zeta_2}) = O(\kappa^{3-i})$$

and

$$\partial_\kappa^i(1-\sqrt{1-2\zeta_1}) = O(\kappa^{1-i}), \quad \partial_\kappa^i(1-\sqrt{1-2\zeta_2}) = O(\kappa^{1-i}).$$

Thus, by (2.27), as $\kappa \to 0$,

$$|\partial_\kappa^4 \widehat{U}_2| = O(\kappa^{-1}), \quad \frac{4}{\kappa}|\partial_\kappa^3 \widehat{U}_2| = O(\kappa^{-1}).$$

Thus, $\Delta^2 \widehat{U}_2$ is integrable and square integrable. And since the $O(\cdot)$ hold uniformly in e on all compact sets, by (2.26),

$$|x|^4 U_2(x) \leqslant \frac{8e^{\frac{3}{2}}}{16e^2} \int \left(\partial_{|k|}^4 + \frac{4}{|k|}\partial_{|k|}^3\right) \hat{U}_2(|k|)\, dk \leqslant \frac{C}{\sqrt{e}}.$$

□