A variant of the Hardy-Littlewood-Sobolev inequality

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Outline

Lorentz space

We begin by recalling/ introducing the Lorentz space.
Let $d \in \mathbb{N}$. Let $|A|$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$.

Definition (Lorentz-(p,q) space $L_{p,q}$)

For $1 \leq p < \infty$, $1 \leq q \leq \infty$, let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. The Lorentz-(p,q) quasi-norm of f is:

$$\text{If } q < \infty, \quad \|f\|^*_{p,q} := \left(p \int_0^\infty \left(\lambda |\{x : |f(x)| > \lambda\}|^{1/p} \right)^q \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/q}$$
$$= \| p^{1/q} \lambda m_f^{1/p} \|_{L^q([0,\infty), \frac{\mathrm{d}\lambda}{\lambda})}.$$
$$\text{If } q = \infty, \quad \|f\|^*_{p,\infty} := \sup_{\lambda > 0} \lambda |\{x : |f(x)| > \lambda\}|^{1/p} = \|\lambda m_f^{1/p}\|_{L^\infty([0,\infty))}.$$

where $m_f(\lambda) \equiv |\{x : |f(x)| > \lambda\}|$.
A measurable function $f \in L_{p,q}$ if $\|f\|^*_{p,q} < \infty$.

- The Lorentz-(p,q) quasi-norm satisfies $\|f + g\|^*_{p,q} \leq 2^{1/p}(\|f\|^*_{p,q} + \|g\|^*_{p,q})$.
- $L_{p,p} = L^p$ and $L_{p,\infty} = L^p_w$.

Facts about Lorentz space

Proposition 1 (Embedding of $L_{p,q}$ spaces in the 2nd index)

If $1 \leq p < \infty$, $1 \leq q < r \leq \infty$, *then*

$$\|f\|_{p,r}^* \leq C\|f\|_{p,q}^*$$

where $C = C(p,q,r) = (\frac{q}{p})^{1/q-1/r} > 0$. *In particular,* $L_{p,1} \subset L_{p,p} = L^p \subset L_{p,\infty}$.

In contrast to the L^p space, the space is smaller when the second index gets smaller.

Proposition 2 (Existence of equivalent norm for $p \neq 1$)

For $1 < p < \infty$, $1 \leq q < r \leq \infty$ *(i.e.* $p \neq 1$*), there exists an equivalent norm* $\|\cdot\|_{p,q}$ *such that*

$$\|f\|_{p,q}^* \leq \|f\|_{p,q} \leq \frac{p}{p-1}\|f\|_{p,q}^*. \tag{1}$$

Moreover, the norm $\|f\|_{1,q}$ *exists and* $\|f\|_{1,q}^* \leq \|f\|_{1,q}$ *is true.*

Definitions for Theorem 8.2

Definition (Operator $\mathcal{G}_\beta$)

For $0 < \beta < d$, define the operator $\mathcal{G}_\beta$:

$$\mathcal{G}_\beta f(x) = (|\cdot|^{-\beta} * f)(x) = \int_{\mathbb{R}^d} |x-y|^{-\beta} f(y)\, \mathrm{d}y$$

Lemma 3 (Hardy-Littlewood-Sobolev inequality)

Let $p, q > 1$, $0 < \beta < d$ with $1/p + 1/q + \beta/d = 2$. Let $f \in L^p, h \in L^q$. Then there exists a constant $C = C(d, \beta, p)$ such that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{-\beta} h(y) \, \mathrm{d}x\mathrm{d}y \right| \leq C \|f\|_p \|h\|_q.$$

Corollary 4 (q-norm bound of $\mathcal{G}_\beta f$ for $q > d/\beta$)

For $0 < \beta < d$, $1 < p < d/(d-\beta)$, $d/\beta < q < \infty$ with $1/q = 1/p - (d-\beta)/d$. Let $f \in L^p$. Then there exists a constant $C = C(d, \beta, p)$ such that

$$\|\mathcal{G}_\beta f\|_q = \left(\int_{\mathbb{R}^d} \left| |x-y|^{-\beta} f(y) \right|^q \mathrm{d}y \right)^{1/q} \leq C \|f\|_p$$

Proof of Corollary 4: Choose h in Lemma 3 such that $(\mathcal{G}_\beta f)(y)h(y) = |\mathcal{G}_\beta f(y)|^{q'}$ where $1/q + 1/q' = 1$. Then $|h| = |\mathcal{G}_\beta f|^{q'/q}$ and

$$\|\mathcal{G}_\beta f\|_{q'}^{q'} \leq C \|f\|_p \|\mathcal{G}_\beta f\|_{q'}^{q'/q}.$$

Extract the conditions for q' in Lemma 3 and rename q' by q. □

Definitions for Theorem 8.2 (2)

Definition ($\mathcal{L}_{\beta,s}$ space, Def 8.1)

Let $0 < \beta < d$, $s > d$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $(1 + |x|)^s f(x) \in L^1$ and $f \in L_{d/(d-\beta),1}$. Denote $f_{\leq R} := f \cdot \mathbb{1}_{\overline{B_R}}$ and $f_{>R} := f - f_{\leq R}$. Define

$$\|\!|f|\!\|_{\beta,s} := \int_{\mathbb{R}^d} (1 + |x|)^{s-d} |f(x)| dx + \sup_{R>0} (1 + R)^{\beta+s-d} \|f_{>R}\|_{d/(d-\beta),1}. \quad (2)$$

A measurable function $f \in \mathcal{L}_{\beta,s}$ if $\|\!|f|\!\|_{\beta,s} < \infty$.

Assume $f \in \mathcal{L}_{\beta,s}$, then

1. from the definition of $\mathcal{L}_{\beta,s}$-norm, $\|f\|_1 \leq \|\!|f|\!\|_{\beta,s}$ and $\|f\|_{d/(d-\beta),1} \leq \|\!|f|\!\|_{\beta,s}$.
2. Since $d/(d - \beta) > 1$, by embedding, we have $f \in L_{d/(d-\beta),1} \subset L_{d/(d-\beta),d/(d-\beta)} = L^{d/(d-\beta)}$. Thus, $f \in L^1 \cap L^{d/(d-\beta)}$.
3. By interpolation, $f \in L^p$, for $1 \leq p \leq d/(d - \beta)$.

Example of $\mathcal{L}_{\beta,s}$ function, Lem 8.6

Suppose there exists $M > 0$, $r > 0$ such that $|f(x)| \leq M(1 + |x|)^{-r} \quad \forall x \in \mathbb{R}^d$.
Then for all $s > 0$ such that $d < s < r$, for all $\beta > 0$, there exists $C = C(r - s, d, \beta)$ such that

$$\|f\|_{\beta,s} \leq CM.$$

We will prove this claim after the proof of Theorem 8.2.

Statement of Theorem 8.2

Theorem 5 (q-norm bound of $\mathcal{G}_\beta f$ for some $q \le d/\beta$,Thm 8.2)

Let $f \in \mathcal{L}_{\beta,s}$ for $0 < \beta < d < s \le d+1$ satisfying $\int_{\mathbb{R}^d} f(x)dx = 0$. Then for all $1 \le q \leqslant d/\beta$ such that $q > \frac{d}{\beta+s-d}$, there is a constant C depending only on q, s and β such that

$$\|\mathcal{G}_\beta f\|_q \leqslant C |\!|\!| f |\!|\!|_{\beta,s} \tag{3}$$

Furthermore, for all $1 < p < \infty$, there is a constant C depending only on p, q, s and β such that

$$\|\mathcal{G}_\beta f\|_q \leqslant C \left(\|f\|^{1-\theta}_{d/(d-\beta/p')} |\!|\!| f |\!|\!|^{\theta}_{\beta,s} \right) \tag{4}$$

where

$$\theta := \frac{dp - \beta q}{qp} \frac{p'}{\beta + p'(s-d)} \ , \quad p' := \frac{p}{p-1} \tag{5}$$

Remark: For $1 < p < \infty$, $1 < \frac{d}{d - \frac{\beta}{p'}} < \frac{d}{(d-\beta)}$. If $f \in \mathcal{L}_{\beta,s}$, then $f \in L^p$, for $1 \le p \le \frac{d}{(d-\beta)}$. Hence, Eq. 3 follows from Eq. 4.

Preparations for the proof of Theorem 8.2

We will prove two lemmas (Lemma 8.4 & 8.5) that will be used in the proof of Theorem 8.2.

Lemma 6 (Lem 8.4)

Let $f \in \mathcal{L}_{\beta,s}$ for $d < s \leq d+1$ satisfying $\int_{\mathbb{R}^d} f \,\mathrm{d}x = 0$. Then for a universal constant C,

$$|\mathcal{G}_\beta(f_{\leqslant |x|/2})(x)| \leqslant C \| f \|_{\beta,s} |x|^{-(\beta+s-d)} .$$

where $C = \frac{3}{2}$.

Proof: Let $R > 0$. *Step 1:* Write $|\mathcal{G}_\beta f_{\leq R}|$ using $f_{\leq R}$ and $f_{>R}$.

$$\begin{aligned} |\mathcal{G}_\beta f_{\leq R}(x)| &= \left| \int |x-y|^{-\beta} f_{\leq R}(y) \,\mathrm{d}y \right| \\ &= \left| \int (|x-y|^{-\beta} - |x|^{-\beta} + |x|^{-\beta}) f_{\leq R}(y) \,\mathrm{d}y \right| \\ &= \left| \int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\leq R}(y) \,\mathrm{d}y + |x|^{-\beta} \underbrace{\int f_{\leq R}(y) \,\mathrm{d}y}_{= -\int f_{>R}(y) \,\mathrm{d}y} \right| \end{aligned}$$

Proof of $|\mathcal{G}_\beta(f_{\leqslant|x|/2})(x)| \leqslant C\|\!|f|\!\|_{\beta,s}|x|^{-(\beta+s-d)}$.

$$|\mathcal{G}_\beta f_{\leq R}(x)| = \left| \int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\leq R}(y)\,\mathrm{d}y + |x|^{-\beta} \underbrace{\int f_{\leq R}(y)\,\mathrm{d}y}_{=-\int f_{>R}(y)\,\mathrm{d}y} \right|$$

$$\leq \left| \int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\leq R}(y)\,\mathrm{d}y \right| + \left| -|x|^{-\beta} \int f_{>R}(y)\,\mathrm{d}y \right|. \quad (6)$$

Step 2: To estimate the first term, we use the Fundamental theorem of calculus (HDI). For $x, y \in \mathbb{R}^d$,

$$|x+y|^{-\beta} - |x|^{-\beta} = \int_0^1 \frac{d}{dt}(|x+ty|^2)^{-\beta/2}\,\mathrm{d}t.$$

Since

$$\frac{d}{dt}(|x+ty|^2)^{-\beta/2} = -\beta(|x+ty|^2)^{-\frac{\beta+2}{2}}(x+ty)\cdot y,$$

we have

$$\int_0^1 \frac{d}{dt}(|x+ty|^2)^{-\beta/2}\,\mathrm{d}t = -\beta \int_0^1 (x\cdot y + t|y|^2)(|x+ty|^2)^{-\frac{(\beta+2)}{2}}\,\mathrm{d}t.$$

$$||x-y|^{-\beta}-|x|^{-\beta}| = \int_0^1 \frac{d}{dt}(|x+ty|^2)^{-\beta/2}\,\mathrm{d}t = -\beta\int_0^1 (|x+ty|^2)^{-\frac{(\beta+2)}{2}}(x\cdot y+t|y|^2)\mathrm{d}t.$$

For $t \in [0,1]$,

$$|x\cdot y + t|y|^2| \leq |x||y| + t|y|^2 \leq |x||y| + |y|^2.$$

In addition, we require $2R \leq |x|$. Recall

$$||\mathcal{G}_\beta f_{\leq R}(x)| \leq \left|\int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\leq R}(y)\,\mathrm{d}y\right| + \left|-|x|^{-\beta}\int f_{>R}(y)\,\mathrm{d}y\right|. \quad (6)$$

In 6, the first integral is over y such that $|y| \leq R$. Under the condition $2R \leq |x|$, $|y| \leq |x|$ and we have

$$|x+ty|^2 \overset{\text{Jensen}}{\leq} 2(|x|^2 + t^2|y|^2) \leq 2(|x|^2+|y|^2) \leq (2|x|)^2.$$

By the above estimates,

$$\begin{aligned} ||x-y|^{-\beta} - |x|^{-\beta}| \leq& \beta\int_0^1 ((2|x|)^2)^{-\frac{(\beta+2)}{2}}(|x||y|+|y|^2)\,\mathrm{d}t \\ =& \beta 2^{-(\beta+2)}|x|^{-\beta-2}(|x||y|+|y|^2). \end{aligned}$$

Hence the first term in Eq. 6 can be bounded by:

$$\left|\int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\le R}(y)\ \mathrm{d}y\right| \le \int \left||x-y|^{-\beta} - |x|^{-\beta}\right| |f_{\le R}(y)|\ \mathrm{d}y$$
$$\le \beta 2^{-(\beta+2)} |x|^{-\beta-2} \int (|x||y| + |y|^2) |f_{\le R}(y)|\ \mathrm{d}y.$$

With $0 < s - d \le 1$ and $|y| \le R$, we have

$$|y| \le |y| (\frac{1+|y|}{|y|})^{s-d} \underset{0\le 1+d-s<1}{\le} R^{1+d-s}(1+|y|)^{s-d},$$

and

$$|y|^2 \le |y|^2 (\frac{1+|y|}{|y|})^{s-d} \underset{1\le 2+d-s<2}{\le} R^{2+d-s}(1+|y|)^{s-d}.$$

Therefore, for $j = 1$ or 2 and use the definition of $\|\|f\|\|_{\beta,s}$ Eq. 2:
$\|\|f\|\|_{\beta,s} := \int_{\mathbb{R}^d} (1+|x|)^{s-d} |f(x)| dx + \sup_{R>0} (1+R)^{\beta+s-d} \|f_{>R}\|_{d/(d-\beta),1}$,

$$\int (|y|^j) |f_{\le R}(y)|\ \mathrm{d}y \le R^{j+d-s} \int (1+|y|)^{s-d} |f(y)|\ \mathrm{d}y \le R^{j+d-s} \|\|f\|\|_{\beta,s}.$$

$$\|\mathcal{G}_\beta f_{\leq R}(x)| \leq \left|\int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\leq R}(y)\, \mathrm{d}y\right| + \left|-|x|^{-\beta}\int f_{>R}(y)\, \mathrm{d}y\right|. \quad (6)$$

Step 3: To estimate the second term in Eq. 6, we observe, for $s > d$,

$$\frac{(1+|y|)^{s-d}}{R^{s-d}} \geq 1 \quad \text{if } |y| \geq R,$$

$$\frac{(1+|y|)^{s-d}}{R^{s-d}} \geq \frac{1}{R^{s-d}} \geq 0 \quad \text{if } |y| < R.$$

Then

$$|f_{>R}(y)| = \mathbb{1}_{\overline{B_R}^c}(y)|f(y)| \leq \frac{(1+|y|)^{s-d}}{R^{s-d}}|f(y)|.$$

By the definition of $\|f\|_{\beta,s}$ Eq. 2:
$\|f\|_{\beta,s} := \int_{\mathbb{R}^d}(1+|x|)^{s-d}|f(x)|dx + \sup_{R>0}(1+R)^{\beta+s-d}\|f_{>R}\|_{d/(d-\beta),1}$, we have

$$\int |f_{>R}(y)|\, \mathrm{d}y \leq R^{d-s}\int (1+|y|)^{s-d}|f(y)|\, \mathrm{d}y \leq R^{d-s}\|f\|_{\beta,s}.$$

Final step: Return to the main argument,

$$\begin{aligned} |\mathcal{G}_\beta f_{\le R}(x)| \le & \left| \int (|x-y|^{-\beta} - |x|^{-\beta}) f_{\le R}(y) \, \mathrm{d}y \right| + \left| -|x|^{-\beta} \int f_{>R}(y) \, \mathrm{d}y \right| \qquad (6) \\ \le & \beta 2^{-(\beta+2)} |x|^{-\beta-2} \int (|x||y| + |y|^2) |f_{\le R}(y)| \, \mathrm{d}y + |x|^{-\beta} \int |f_{>R}(y)| \, \mathrm{d}y \\ \le & \beta 2^{-(\beta+2)} |x|^{-\beta-2} (|x| R^{1+d-s} \|f\|_{\beta,s} + R^{2+d-s} \|f\|_{\beta,s}) + |x|^{-\beta} R^{d-s} \|f\|_{\beta,s} \\ = & \|f\|_{\beta,s} \left(\frac{1}{|x|^\beta R^{s-d}} + \frac{\beta 2^{-(\beta+2)} R^{1+d-s}}{|x|^{\beta+1}} + \frac{\beta 2^{-(\beta+2)} R^{2+d-s}}{|x|^{2+\beta}} \right). \end{aligned}$$

Now fix $R = |x|/2$ to get

$$|\mathcal{G}_\beta f_{\le |x|/2}(x)| \le \|f\|_{\beta,s} |x|^{-(\beta+s-d)} (1 + 2\beta 2^{-(\beta+2)}).$$

Since the maximum of the map $\beta \mapsto \beta 2^{-\beta}$ is attained at $\beta = \frac{1}{\log(2)}$ with value $\frac{e^{-1}}{\log(2)} \le 1$. We can take $C = 1 + 1/2 = 3/2$ to get

$$|\mathcal{G}_\beta f_{\le |x|/2}(x)| \le C \|f\|_{\beta,s} |x|^{-(\beta+s-d)}.$$

□

Lemma 8.5

Lemma 7 (Lem 8.5)

Let $f \in \mathcal{L}_{\beta,s} \subset L^{d/(d-\beta)}$. Then for a universal constant C,

$$|\mathcal{G}_\beta(f_{>|x|/2})(x)| \leqslant C \|\!| f \|\!|_{\beta,s} (1 + |x|)^{-(\beta+s-d)} \ .$$

Ingredients in the proof of Lemma 8.5

Lemma 8 (O'Neil's extension of Hölder's inequality)

For $1 \le p, q \le \infty$, if $f \in L_{p,q}$, $g \in L_{p',q'}$, where $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$ then $fg \in L^1$and

$$\|fg\|_1 \le M\|f\|_{p,q}\|g\|_{p',q'}.$$

where M is a universal constant.

Proposition 9

For $d > \beta > 0$, the function $|x|^{-\beta}$ is in $L_{d/\beta,\infty}$ and

$$\||x|^{-\beta}\|_{d/\beta,\infty} \le \frac{8\pi^2}{15}.$$

Proof:

$$\begin{aligned} \sup_{\lambda>0} \lambda|\{x : |x|^{-\beta} > \lambda\}|^{\beta/d} &= \sup_{\lambda>0} \lambda\lambda^{-\frac{d}{\beta}\frac{\beta}{d}}|\{x : |x| < 1\}|^{\beta/d} \\ &\le |\{x : |x| < 1\}| \le |B_5| = \frac{8\pi^2}{15}. \end{aligned}$$

Proof of Lemma 8.5

[Lem 8.5] Let $f \in \mathcal{L}_{\beta,s}$. Then for a universal constant C, $|\mathcal{G}_\beta(f_{>|x|/2})(x)| \leqslant C\|f\|_{\beta,s}(1+|x|)^{-(\beta+s-d)}$.

Proof:Let $R > 0$. We have

$$
\begin{aligned}
|\mathcal{G}_\beta f_{>R}(x)| &= \left| \int |x-y|^\beta |f_{>R}(y)| \, \mathrm{d}y \right| \\
&\overset{O'Neil}{\leq} M \| |x-\cdot|^{-\beta} \|_{d/\beta,\infty} ||f_{>R}||_{d/(d-\beta),1} \\
&= M \sup_{\lambda>0} \lambda |\{y : |x-y|^{-\beta} > \lambda\}|^{\beta/d} ||f_{>R}||_{d/(d-\beta),1} \\
&\overset{\bigstar}{=} M \sup_{\lambda>0} \lambda |\{y : |y|^{-\beta} > \lambda\}|^{\beta/d} ||f_{>R}||_{d/(d-\beta),1} \\
&= M \| |y|^{-\beta} \|_{d/\beta,\infty} ||f_{>R}||_{d/(d-\beta),1} \\
&\leq C_M ||f_{>R}||_{d/(d-\beta),1} \quad \forall x \in \mathbb{R}^d,
\end{aligned}
$$

where $C_M =: M\frac{8\pi^2}{15}$. We used Lebesgue measure is translational invariant at $\bigstar$.

Proof of $|\mathcal{G}_\beta(f_{>|x|/2})(x)| \leqslant C\|\!|f|\!\|_{\beta,s}(1+|x|)^{-(\beta+s-d)}$

By definition of $\|\!|f|\!\|_{\beta,s}$ Eq. 2:
$\|\!|f|\!\|_{\beta,s} := \int_{\mathbb{R}^d}(1+|x|)^{s-d}|f(x)|dx + \sup_{R>0}(1+R)^{\beta+s-d}\|f_{>R}\|_{d/(d-\beta),1}$, for any $R>0$,

$$(1+R)^{\beta+s-d}\|f_{>R}\|_{d/(d-\beta),1} \leq \|\!|f|\!\|_{\beta,s},$$

$$\|f_{>R}\|_{d/(d-\beta),1} \leq (1+R)^{-(\beta+s-d)}\|\!|f|\!\|_{\beta,s}.$$

Then

$$|\mathcal{G}_\beta f_{>R}(x)| \leq C_M||f_{>R}||_{d/(d-\beta),1} \leq C_M(1+R)^{-(\beta+s-d)}\|\!|f|\!\|_{\beta,s}.$$

Choose $R=|x|/2$ to get

$$|\mathcal{G}_\beta f_{>|x|/2}(x)| \leq C_M(1+\frac{|x|}{2})^{-(\beta+s-d)}\|\!|f|\!\|_{\beta,s} \leq C_M(1+|x|)^{-(\beta+s-d)}\|\!|f|\!\|_{\beta,s},$$

with $C_M =: M\frac{8\pi^2}{15}$. $\square$

Proof of Theorem 8.2

Theorem 5 (q-norm bound of $\mathcal{G}_\beta f$ for some $q \leq d/\beta$,Thm 8.2)

Let $f \in \mathcal{L}_{\beta,s}$ for $0 < \beta < d < s \leq d+1$ satisfying $\int_{\mathbb{R}^d} f(x)dx = 0$. Then for all $1 \leq q \leqslant d/\beta$ such that $q > \frac{d}{\beta+s-d}$, there is a constant C depending only on q, s and β such that

$$\|\mathcal{G}_\beta f\|_q \leqslant C \|\!|f|\!\|_{\beta,s} \tag{3}$$

Furthermore, for all $1 < p < \infty$, there is a constant C depending only on p, q, s and β such that

$$\|\mathcal{G}_\beta f\|_q \leqslant C \left(\|f\|_{d/(d-\beta/p')}^{1-\theta} \|\!|f|\!\|_{\beta,s}^{\theta} \right) \tag{4}$$

where

$$\theta := \frac{dp - \beta q}{qp} \frac{p'}{\beta + p'(s-d)} \ , \quad p' := \frac{p}{p-1} \tag{5}$$

Proof: Let $1 \leq q \leq d/\beta$ and $R > 0$. We can write

$$\mathcal{G}_\beta f = (\mathcal{G}_\beta f)_{\leq R} + (\mathcal{G}_\beta f)_{>R}.$$

Proof of $\|\mathcal{G}_\beta f\|_q \leqslant C\left(\|f\|_{d/(d-\beta/p')}^{1-\theta}\|\!|f|\!\|_{\beta,s}^{\theta}\right)$

$$\mathcal{G}_\beta f = (\mathcal{G}_\beta f)_{\leq R} + (\mathcal{G}_\beta f)_{>R}.$$

$$\begin{aligned}
\int |\mathcal{G}_\beta f|^q \,\mathrm{d}x &= \int |(\mathcal{G}_\beta f)_{\leq R} + (\mathcal{G}_\beta f)_{>R}|^q \mathrm{d}x \\
&= \int_{\leq R} |(\mathcal{G}_\beta f)_{\leq R}|^q \,\mathrm{d}x + \int_{>R} |(\mathcal{G}_\beta f)_{>R}|^q \,\mathrm{d}x \\
&= \underbrace{\int |(\mathcal{G}_\beta f)_{\leq R}|^q \,\mathrm{d}x}_{=:(\mathrm{I})} + \underbrace{\int |(\mathcal{G}_\beta f)_{>R}|^q \,\mathrm{d}x}_{=:(\mathrm{II})}.
\end{aligned}$$

$$\int |\mathcal{G}_\beta f|^q \,\mathrm{d}x = \underbrace{\int |(\mathcal{G}_\beta f)_{\le R}|^q \,\mathrm{d}x}_{=:(\mathrm{I})} + \underbrace{\int |(\mathcal{G}_\beta f)_{> R}|^q \,\mathrm{d}x}_{=:(\mathrm{II})}.$$

We consider the first term.

$$\begin{aligned}
(\mathrm{I}) &= \|(\mathcal{G}_\beta f)_{\le R}\|_q^q \overset{\text{Hölder}}{\le} \|\mathcal{G}_\beta f\|_{pd/\beta}^q \|\mathbb{1}_{\le R}\|_{\frac{pdq}{pd-\beta q}}^q \\
&= \|\mathcal{G}_\beta f\|_{pd/\beta}^q |B(0,R)|^{\frac{pd-\beta q}{pdq}\cdot q} = \|\mathcal{G}_\beta f\|_{pd/\beta}^q (|B_d| R^d)^{1-\frac{\beta q}{pd}} \\
&\overset{\text{HLS}}{\le} (C_{\text{HLS}} \|f\|_{\frac{d}{d-\beta/p'}})^q (|B_d| R^d)^{1-\frac{\beta q}{pd}} \\
&\le (C_{\text{HLS}} \|f\|_{\frac{d}{d-\beta/p'}})^q |B_5| R^{d-\frac{\beta q}{p}}
\end{aligned}$$

where $1/(pd/\beta) + 1/(\frac{pdq}{pd-\beta q}) = 1/q$, $1/p + 1/p' = 1$ and $C_{\text{HLS}} = C_{\text{HLS}}(d,\beta,p)$ is the constant from Corollary 4. Note that $d/\beta < pd/\beta < \infty$ and $1 < \frac{d}{d-\beta/p'} < \frac{d}{d-\beta}$ so Corollary 4 applies. The last inequality uses that $0 < \frac{\beta}{pd} \le \frac{\beta q}{pd} \le \frac{1}{p} < 1$ and $|B_d| \le |B_5| = \frac{8\pi^2}{15} \quad \forall d \in \mathbb{N}$.

Lem 8.4: $|\mathcal{G}_\beta(f_{\leqslant |x|/2})(x)| \leqslant C\|f\|_{\beta,s}|x|^{-(\beta+s-d)}$.
Lem 8.5: $|\mathcal{G}_\beta(f_{>|x|/2})(x)| \leqslant C\|f\|_{\beta,s}(1+|x|)^{-(\beta+s-d)}$.
For the second term, we apply Lemma 8.4 and 8.5. It follows from Lemma 8.4 and 8.5 that

$$\begin{aligned} |\mathcal{G}_\beta f(x)| =& |\mathcal{G}_\beta(f_{\leq \frac{|x|}{2}} + f_{> \frac{|x|}{2}})(x)| = |\mathcal{G}_\beta(f_{\leq \frac{|x|}{2}}) + \mathcal{G}_\beta(f_{> \frac{|x|}{2}})(x)| \\ \leq& |\mathcal{G}_\beta f_{\leq \frac{|x|}{2}}(x)| + |\mathcal{G}_\beta f_{> \frac{|x|}{2}}(x)| \\ \leq& \underbrace{C_4\|f\|_{\beta,s}|x|^{-(\beta+s-d)}}_{8.4} + \underbrace{C_5\|f\|_{\beta,s}(1+|x|)^{-(\beta+s-d)}}_{8.5} \\ \overset{\beta+s-d>0}{\leq}& C_{4+5}\|f\|_{\beta,s}|x|^{-(\beta+s-d)}, \end{aligned}$$

where $C_{4+5} := C_4 + C_5$. Therefore, we have

$$\begin{aligned} (\mathrm{II}) = \int |\mathcal{G}_\beta f(x)|^q \mathbb{1}_{>R}(x)\,\mathrm{d}x \leq& \int (C_{4+5}\|f\|_{\beta,s}|x|^{-(\beta+s-d)})^q \mathbb{1}_{>R}(x)\,\mathrm{d}x \\ =& (C_{4+5}\|f\|_{\beta,s})^q\ |B_d| d \int_R^\infty t^{-q(\beta+s-d)} t^{d-1}\,\mathrm{d}t. \end{aligned}$$

$$(\mathrm{II}) \leq (C_{4+5}\|f\|_{\beta,s})^q \; |B_d| d \int_R^\infty t^{-q(\beta+s-d)} t^{d-1} \, \mathrm{d}t.$$

We now need $q > \frac{d}{\beta+s-d} \iff -q(\beta+s-d)+(d-1) < -1$. Then $t^{d-1-q(\beta+s-d)}$ is integrable on $[R,\infty)$ and

$$\int_R^\infty t^{d-1-q(\beta+s-d)} = \frac{1}{q(\beta+s-d)-d} R^{d-q(\beta+s-d)}.$$

Notice that $|B_d|d \leq \frac{16\pi^3}{15} := S_7 \quad \forall d \in \mathbb{N}$ where S_7 is the area of a unit sphere for $d=7$. We have

$$(\mathrm{II}) \leq (C_{4+5}\|f\|_{\beta,s})^q \; S_7 \frac{1}{q(\beta+s-d)-d} R^{d-q(\beta+s-d)}.$$

Collecting the results, we have

$$\begin{aligned}\int |\mathcal{G}_\beta f|^q \, \mathrm{d}x =& (\mathrm{I}) + (\mathrm{II}) \\ \leq& (C_{\mathrm{HLS}}\|f\|_{\frac{d}{d-\beta/p'}})^q |B_5| R^{d-\frac{\beta q}{p}} + (C_{4+5}\|f\|_{\beta,s})^q \; S_7 \frac{R^{d-q(\beta+s-d)}}{q(\beta+s-d)-d}.\end{aligned}$$

$$\int |\mathcal{G}_\beta f|^q \, \mathrm{d}x \leq (C_{\mathrm{HLS}} \|f\|_{\frac{d}{d-\beta/p'}})^q |B_5| R^{d-\frac{\beta q}{p}} + (C_{4+5} \|\!|f|\!\|_{\beta,s})^q \; S_7 \frac{R^{d-q(\beta+s-d)}}{q(\beta+s-d)-d}.$$

To minimise the last expression, we require

$$\|f\|^q_{\frac{d}{d-\beta/p'}} R^{d-\frac{\beta q}{p}} \sim \|\!|f|\!\|^q_{\beta,s} R^{d-q(\beta+s-d)}.$$

Since $q/(d - \frac{\beta q}{p} - (d - q(\beta + s - d))) = \frac{p'}{\beta + p'(s-d)}$, we choose

$$R = \left(\frac{\|\!|f|\!\|_{\beta,s}}{\|f\|_{\frac{d}{d-\beta/p'}}} \right)^{\frac{p'}{\beta+p'(s-d)}} .$$

With this choice of R and some fraction arithmetic, we obtain

$$\int |\mathcal{G}_\beta f|^q \, \mathrm{d}x \leq (C^q_{\mathrm{HLS}} |B_5| + C^q_{4+5} S_7)(\|f\|^{1-\theta}_{\frac{d}{d-\beta/p'}} \|\!|f|\!\|^\theta_{\beta,s})^q,$$

where $\theta := \frac{dp-\beta q}{qp} \frac{p'}{\beta+p'(s-d)}$, $\quad p' := \frac{p}{p-1}$. Taking the q-th root on both side, use $q \geq 1$ and $(a^q + b^q) \leq (a+b)^q$ for $a, b > 0$, we have

$$\|\mathcal{G}_\beta f\|_q \leq (C_{\mathrm{HLS}} |B_5| + C_{4+5} S_7) \|f\|^{1-\theta}_{\frac{d}{d-\beta/p'}} \|\!|f|\!\|^\theta_{\beta,s}.$$

□

Proof of Example/Lemma 8.6

Example of $\mathcal{L}_{\beta,s}$ function, Lem 8.6

Suppose there exists $M > 0$, $r > 0$ such that $|f(x)| \leq M(1+|x|)^{-r} \quad \forall x \in \mathbb{R}^d$. Then for all $s > 0$ such that $d < s < r$, for all $\beta > 0$, there exists $C = C(r-s, d, \beta)$ such that

$$\|\!|f|\!\|_{\beta,s} \leq CM.$$

Proof: Recall

$$\|\!|f|\!\|_{\beta,s} := \int_{\mathbb{R}^d} (1+|x|)^{s-d}|f(x)|dx + \sup_{R\geq 0}(1+R)^{\beta+s-d}\|f_{>R}\|_{d/(d-\beta),1}.$$

We first consider $\|f_{>R}\|^*_{d/(d-\beta),1}$ in the second term of $\|\!|f|\!\|_{\beta,s}$. Let $p' := d/(d-\beta)$ and thus $p = d/\beta$ such that $1/p + 1/p' = 1$. We have

$$\|f_{>R}\|^*_{p',1} := p' \int_0^\infty |\{x : |f_{>R}(x)| > \lambda\}|^{1/p'} \, \mathrm{d}\lambda.$$

$\|f_{>R}\|^*_{p',1} := p' \int_0^\infty |\{x : |f_{>R}(x)| > \lambda\}|^{1/p'} \, \mathrm{d}\lambda.$

For $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$ and $|x| > R$, we have

$$\lambda < |f(x)| \le M(1+|x|)^{-r} < M(1+R)^{-r}$$

and the first two inequalities give

$$|x| < \left(\frac{M}{\lambda}\right)^{1/r}.$$

Moreover

$$\begin{aligned}\{x : |f_{>R}| > \lambda\} \subset & \{x : M(1+R)^{-r} > \lambda\} \\ & \Longrightarrow \{x : |f_{>R}| > \lambda\} \cap \{x : M(1+R)^{-r} \le \lambda\} = \emptyset.\end{aligned}$$

Therefore,

$$|\{x : |f_{>R}(x) > \lambda\}| \begin{cases} = 0 & \text{if } M(1+R)^{-r} \le \lambda \\ \le |\{x : |x| < (\frac{M}{\lambda})^{1/r}\}| = |B_d|(\frac{M}{\lambda})^{d/r} & \text{if } M(1+R)^{-r} > \lambda. \end{cases}$$

In summary,

$$|\{x : |f_{>R}(x)| > \lambda\}| \le \mathbb{1}_{\{\lambda < M(1+R)^{-r}\}}(\lambda)|B_d|(\frac{M}{\lambda})^{d/r}.$$

$$|\{x : |f_{>R}(x)| > \lambda\}| \leq \mathbb{1}_{\{\lambda < M(1+R)^{-r}\}}(\lambda)|B_d|(\frac{M}{\lambda})^{d/r}.$$

Applying the above estimate and note that $r > d \implies r > d - \beta$ for $\beta > 0$ so $p' := d/(d-\beta) > d/r$.

$$\begin{aligned}
\|f_{>R}\|^*_{p',1} :=& p' \int_0^\infty |\{x : |f_{>R}(x)| > \lambda\}|^{1/p'} \, \mathrm{d}\lambda \\
&\leq p' \int_0^{M(1+R)^{-r}} \left(|B_d| \left(\frac{M}{\lambda}\right)^{d/r} \right)^{1/p'} \, \mathrm{d}\lambda \\
&\overset{p'>d/r}{=} p' |B_d|^{1/p'} M^{d/(rp')} \frac{1}{1 - d/(rp')} \left(M(1+R)^{-r}\right)^{-d/(rp')+1} \\
&\overset{p=d/\beta}{=} M |B_d|^{(d-\beta)/d} \frac{d}{d-\beta} \frac{1}{1 - \frac{d}{r}\frac{d-\beta}{d}} (1+R)^{d-\beta-r} \\
&\leq M |B_d| \frac{d}{d-\beta} \frac{1}{1 - \frac{d-\beta}{r}} (1+R)^{d-\beta-r} \\
&\leq M S_7 \frac{1}{d-\beta} \frac{1}{1 - \frac{d-\beta}{r}} (1+R)^{d-\beta-r}
\end{aligned}$$

where $S_7 := \frac{16\pi^3}{15}$ is the area of a unit sphere for $d = 7$.

$$\|f_{>R}\|^*_{p',1} \leq MS_7 \frac{1}{d-\beta} \frac{1}{1-\frac{d-\beta}{r}} (1+R)^{d-\beta-r}.$$

Note that for $r > d - \beta$, the map

$$r \mapsto \frac{1}{1-\frac{d-\beta}{r}} = 1 + \frac{d-\beta}{r-(d-\beta)} > 0$$

is decreasing and the map

$$r \mapsto (1+R)^{-r}$$

is also deceasing on $r \geq 0$. Therefore for all s such that $r > s > d > d - \beta$,

$$\begin{aligned}(1+R)^{\beta+s-d}\|f_{>R}\|^*_{\frac{d}{d-\beta},1} \leq & MS_7 \frac{1}{d-\beta} \frac{1}{1-\frac{d-\beta}{r}} (1+R)^{-(r-s)} \\ \leq & MS_7 \frac{1}{d-\beta} \frac{1}{1-\frac{d-\beta}{d}} (1+R)^{-0} = MS_7 \frac{1}{d-\beta} \frac{d}{\beta}.\end{aligned}$$

By Eq. 1: $\|f\|_{p,q} \leq \frac{p}{p-1}\|f\|^*_{p,q}$, we have $\|f_{>R}\|_{\frac{d}{d-\beta},1} \leq \frac{d}{\beta}\|f_{>R}\|^*_{\frac{d}{d-\beta},1}$. Hence we have the uniform bound over $R > 0$:

$$(1+R)^{\beta+s-d}\|f_{>R}\|_{\frac{d}{d-\beta},1} \leq MS_7 \frac{1}{d-\beta}\left(\frac{d}{\beta}\right)^2 \tag{7}$$

For the first term in
$\|\|f\|\|_{\beta,s} := \int_{\mathbb{R}^d}(1+|x|)^{s-d}|f(x)|dx + \sup_{R\geq 0}(1+R)^{\beta+s-d}\|f_{>R}\|_{d/(d-\beta),1}$, we use the assumption $|f(x)| \leq M(1+|x|)^{-r}$ and $r - s > 0$,

$$\int (1+|x|)^{s-d}|f(x)|\ \mathrm{d}x \leq M \underbrace{\int \frac{1}{(1+|x|)^{d+(r-s)}}\ \mathrm{d}x}_{:=C_{r-s,d}<+\infty} = MC_{r-s,d}. \tag{8}$$

Combining the two estimates Eq. 7 and Eq.8, we have

$$\begin{aligned}\|\|f\|\|_{\beta,s} :=& \int_{\mathbb{R}^d}(1+|x|)^{s-d}|f(x)|dx + \sup_{R\geq 0}(1+R)^{\beta+s-d}\|f_{>R}\|_{d/(d-\beta),1}\\ \leq& M\left(C_{r-s,d} + S_7\frac{1}{d-\beta}\left(\frac{d}{\beta}\right)^2\right).\end{aligned}$$

□

More on the extension of Hölder's inequality

The goal of this section is to provide some ideas to prove Lemma 8. However, our approach of proof require materials on rearrangement functions that are not covered in today's seminar. Statements are given in this section without proofs. For more details, see [Simon, 2015][Chapter 2.2, Chapter 6.1].

Definition (Decreasing rearrangement)

Let f a measurable function from a σ-finite measure space (M, Σ, μ) to $\mathbb{R}$ (or $\mathbb{C}$) which is finite μ-almost everywhere and has $\mu(x : |f(x)| > t) < \infty$ for all $t > 0$. Define the decreasing rearrangement of f, $f^* : [0, \infty) \to \mathbb{R}$ as

$$f^*(t) := \inf\{\alpha : m_f(\alpha) \leq t\}$$

with the distribution function of f

$$m_f(\alpha) := \mu(\{x \in M : |f(x)| > \alpha\})$$

Proposition 10 (Properties of decreasing rearrangement)

Let f^* be a decreasing rearrangement of f. Then

1. f^* *is monotone decreasing, i.e.* $t > s \implies f^*(t) \leq f^*(s)$.
2. f^* *is right continuous, i.e.* $\lim_{t \downarrow t_0} f^*(t) = f^*(t_0)$.
3. $\lim_{t \to \infty} f^*(t) = 0$.
4. f^* *is equimeasurable with f, i.e.*

$$\forall\ \alpha > 0 \quad \mu(\{x \in M : |f(x)| > \alpha\}) = |\{t \in [0, \infty) : f^*(t) > \alpha\}|.$$

5. f^* *is the unique function that satisfies (i)-(iv).*

Definition (Running averages of decreasing rearrangement)

Let f^* be the decreasing rearrangement of f. Define the running averages of decreasing rearrangement of f, $f^{**} : (0, \infty) \to \mathbb{R}$ as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, \mathrm{d}s \equiv \langle f^* \rangle_{[0,t]}.$$

Proposition 11

Let f^{**} be the running averages of decreasing rearrangement of f. Then

$$f^* \leq f^{**}.$$

Proposition 12

Let f^{**} be the running averages of decreasing rearrangement of f. Then

$$f^{**}(t) = \sup\{\frac{1}{t} \int_E |f(x)| \, \mathrm{d}\mu(x) : \mu(E) \leq t\}.$$

The equality in Proposition 12 can be seen by the monotone decreasing property and the equimeasurability with f of f^*.

Proposition 13 (Lorentz-(p,q) (quasi)norm in terms of f^* and f^{**})

Let f^* be the decreasing rearrangement of f. For $1 \le p < \infty$, $1 \le q < \infty$,

$$\|f\|^*_{p,q} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q} = \|t^{1/p} f^*\|_{L^q([0,\infty),\frac{\mathrm{d}t}{t})}.$$

For $1 \le p < \infty$, $q = \infty$,

$$\|f\|^*_{p,\infty} = \sup\{t \ge 0 : t^{1/p} f^*(t)\} = \|t^{1/p} f^*\|_{L^\infty([0,\infty))}.$$

Moreover, if we replace f^* by f^{**} on $(0,\infty)$ above, we get a norm on $L_{p,q}$.

- The equality with $\|f\|^*_{p,q}$ defined earlier can be proved by a change of variables and the fact that f^* and f are equimeasurable.
- The triangle inequality of the norm defined with f^{**} can be seen from the characterisation of $f^{**}(t) = \sup\{\frac{1}{t}\int_E |f(x)| \, \mathrm{d}\mu(x) : \mu(E) \le t\}$. Hence $(f+g)^{**} \le f^{**} + g^{**}$ and use the triangle inequality for the L^q norm to conclude.

Proposition 14 (Rearrangement inequality)

Let f, g be measurable on (X, μ), f^*, g^* be their decreasing rearrangements. Then

$$\left|\int_X fg \, \mathrm{d}\mu\right| \leq \int_0^\infty f^*(t)g^*(t) \, \mathrm{d}t$$

The proof of Proposition 14 uses the Wedding Cake Representation, Tonelli's theorem and for any $\alpha, \beta > 0$,

$$\mu(\{|f| > \alpha\} \cap \{|g| > \beta\}) \leq |\{f^* > \alpha\} \cap \{g^* > \beta\}|.$$

Corollary 15

Let f, g be measurable on (X, μ). Then

$$(fg)^{**}(t) = \sup\{\frac{1}{t}\int_E |fg| \, \mathrm{d}\mu : \mu(E) \leq t\} \leq \frac{1}{t}\int_0^t f^*(s)g^*(s) \, \mathrm{d}s.$$

Lemma 8 (O'Neil's extension of Hölder's inequality)

For $1 \le p, q \le \infty$, *if* $f \in L_{p,q}$, $g \in L_{p',q'}$, *where* $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$ *then* $fg \in L^1$ *and*

$$\|fg\|_1 \le M \|f\|_{p,q} \|g\|_{p',q'}.$$

where $M = 1$.

Proof of Lemma 8:

$$\|fg\|_1 = \int |fg| \, \mathrm{d}\mu \overset{\text{equimear.}}{=} \int_0^\infty (fg)^*(s) \, \mathrm{d}s$$

$$\overset{\text{dominated conver.}}{=} \lim_{t\to\infty} \int_0^t (fg)^*(s) \, \mathrm{d}s \overset{\text{def. } (fg)^{**}}{=} \lim_{t\to\infty} t \, (fg)^{**}(t)$$

$$\overset{\text{Cor. 15}}{\le} \lim_{t\to\infty} \int_0^t f^*(s) g^*(s) \, \mathrm{d}s = \int_0^\infty f^*(s) s^{1/p} g^*(s) s^{1/p'} \, \frac{\mathrm{d}s}{s}$$

$$\overset{\text{Hölder}}{\le} \|f^* s^{1/p}\|_{L^q([0,\infty), \frac{\mathrm{d}s}{s})} \, \|g^* s^{1/p'}\|_{L^{q'}([0,\infty), \frac{\mathrm{d}s}{s})} = \|f\|^*_{p,q} \|g\|^*_{p',q'}$$

$$\overset{f^* \le f^{**}}{\le} \|f\|_{p,q} \|g\|_{p',q'},$$

where $1/p + 1/p' = 1 = 1/q + 1/q'$. $\square$

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Questions?