# ON THE CONVOLUTION INEQUALITY

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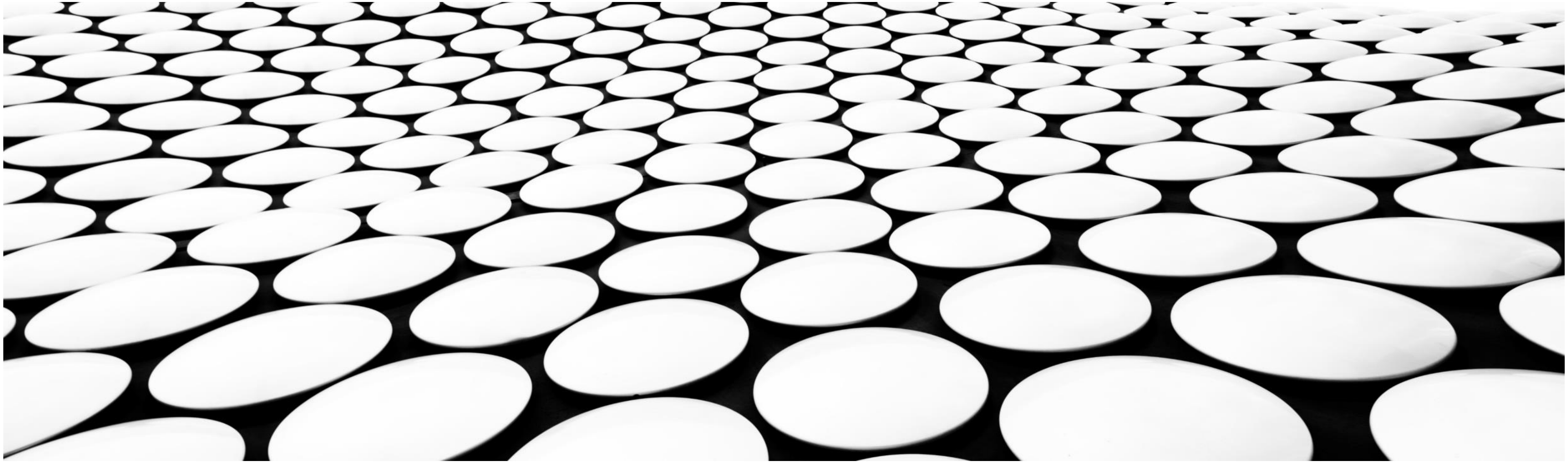
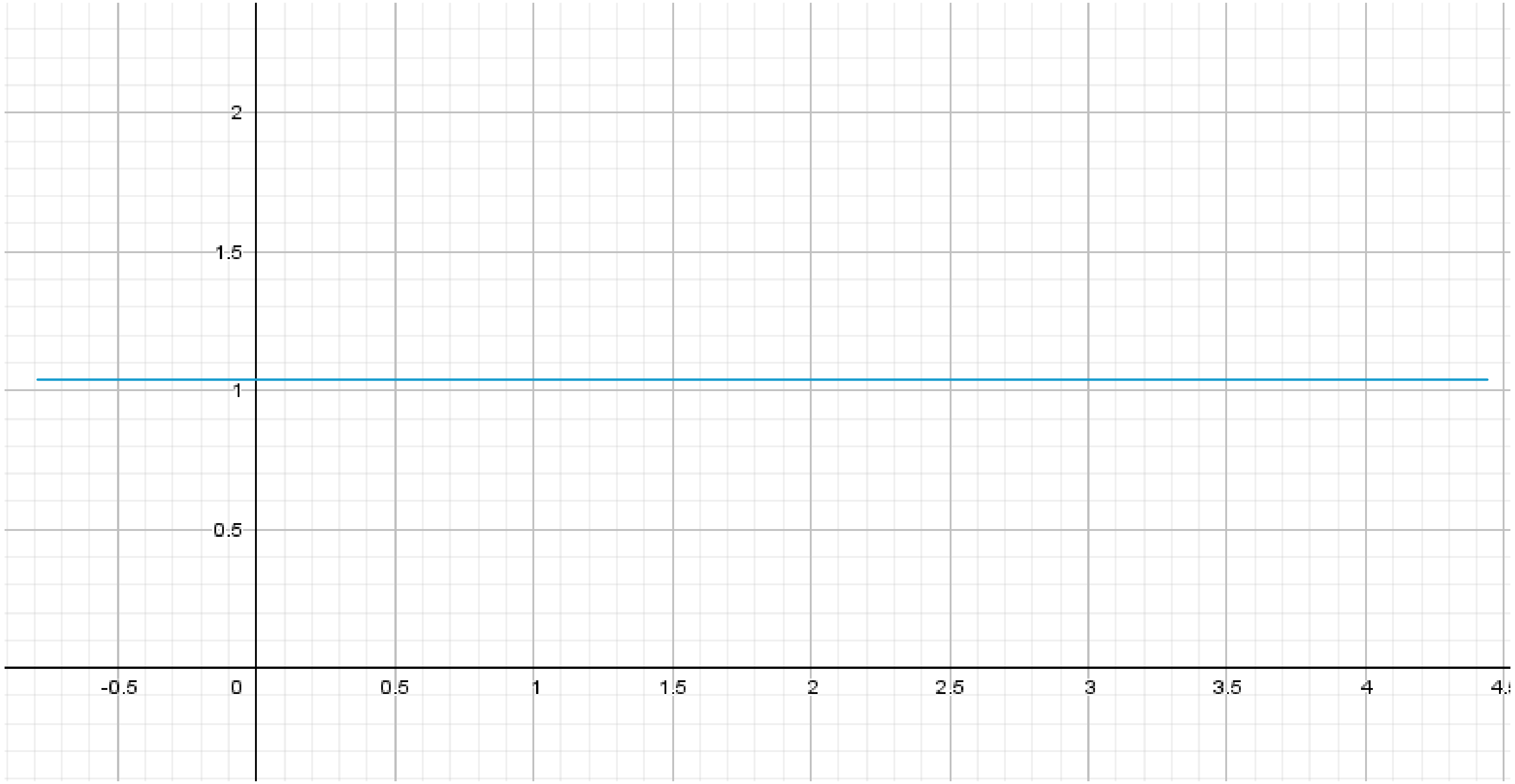


# WHAT ARE THESE FUNCTIONS WE ARE LOOKING AT?

$$f(x) \geq f * f$$




*There are examples in every dimension, f.e indicator function*

*Important, it needs to be defined on an intervall with lenght 2a*

# INTRODUCTION

$$f(x) \geq f * f$$

Element of $L^{\frac{p}{2-p}}(R^d)$ for all p ∈ $[1; 2]$

→ LP-space

BUT: p=1 is special

So, we only consider p=1

# BUT WHAT CHARACTERISTICS ARE INTERESTING?

- Theorem 1: Finding an upper bound for p=1, positivity, finding a general formula for f
- Theorem 2:  Showing that f decays fairly slowly for all these functions with sharp upper bound
- Theorem 4: rapid decay $\int |x|^p f(x)\ dx < \infty.$ for a set of these functions without sharp upper bound

# INTRODUCTION

$$f(x) \geq f * f$$

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BUT: p=1 is special

So, we only consider p=1

(Young's Inequality). *Let* $p, q, r \in [1, \infty]$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

*If* $f \in L^p$ *and* $g \in L^q$ *then* $|f| * |g|(x) < \infty$ *for* $m$ − *a.e.* $x$ *and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

But why is f*f an element of $L^{\frac{p}{2-p}}(R^d)$ for all p ∈ [1; 2]?

$p = q$ ⇒ $\frac{1}{p} + \frac{1}{q} = \frac{2}{p}$

$$\left(\frac{2}{p} - 1\right)^{-1} = r = \left(\frac{p}{2-p}\right)$$

**Theorem 1.** *Let $f$ be a real valued function in $L^1(\mathbb{R}^d)$ such that*

$$f(x) - f \star f(x) =: u(x) \geqslant 0 \tag{5}$$

*for all $x$. Then $\int_{\mathbb{R}^d} f(x)\, dx \leqslant \frac{1}{2}$, and $f$ is given by the convergent series*

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x) \tag{6}$$

*where the $c_n \geqslant 0$ are the Taylor coefficients in the expansion of $\sqrt{1-x}$*

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n, \quad c_n = \frac{(2n-3)!!}{2^n n!} \sim n^{-3/2} \tag{7}$$

*In particular, $f$ is positive. Moreover, if $u \geqslant 0$ is any integrable function with $\int_{\mathbb{R}^d} u(x) \mathrm{d}x \leqslant \frac{1}{4}$, then the sum on the right in (6) defines an integrable function $f$ that satisfies (5).*

# FINDING AN UPPER BOUND FOR $f \geqslant f \star f$

By integration we find:

$q^2 \leq q, if$
$0 \leq q \leq 1$

$$\int_{R^d} f(x) \leq 1$$

Goal: Finding a sharp upper bound!

But why is indeed 0,5 a sharp upper bound?

$$\int_{R^d} f(x) \leq 0{,}5$$

# THEOREM 1

- Only consider real valued function in $L^1(R^d)$

$$\rightarrow \int R^d |f| dx < \infty$$

- Define $f(x) - f * f(x) \equiv u(x) \geq o$

u is integrable!

“The convolution of f and g exists, if f and g are both Lebesgue integrable functions in $L^1(R^d)$, and in this case f∗g is also integrable” [1]

Also:

$$\int_\Omega (\alpha f + \beta g)\, \mathrm{d}\mu = \alpha \cdot \int_\Omega f\, \mathrm{d}\mu + \beta \cdot \int_\Omega g\, \mathrm{d}\mu$$

[1]Stein, Elias; Weiss, Guido (1971), Introduction to Fourier Analysis on Euclidean Spaces, Theorem^1.3

# THEOREM 1: MAKE SOME HELPFUL DEFINITIONS

- Define: $a \equiv \int_{R^d} f(x)dx \; and \; b \equiv \int_{R^d} u(x)dx$

  Obviously, $b \equiv \int_{R^d} u(x)dx \geq 0$

- Fouriertransformation of f $\tilde{f}$ for all $p \in [1;2\,]$, so $f(x) - f * f(x) \equiv u(x)$ becomes

  | |
  |---|
  | Definition: f= $\int_I \; dxe^{-2i\pi kx} f(x) \quad \in \quad L^{\frac{p}{p-1}}(R^d))$ |

- If $f = f * f, then \; \widetilde{f} = \tilde{f}^2$

Only consider equality!

$$\int_{R^d} f(x)dx = 1$$

# THEOREM 1

Change order of variables: $f = u + f * f$

By Fouriertransformation, it follows that

$$\tilde{f}(k) = \tilde{f}^2(k) + \tilde{u}(k)$$

How can we proceed from there? Take: k=0 and use definitions we made

b is positive!
complete the square

So, in the end, we get: $a^2 - a = -b$ $\Rightarrow$ $\left(a - \frac{1}{2}\right)^2 = \frac{1}{4} - b,$

$$a^2 - a + \frac{1}{4} = \frac{1}{4} - b$$

From there, it follows that $0 \leq b \leq \frac{1}{4}$

-a is equal to one!

1-1+0,25=0,25

# THEOREM 1: WHAT CAN WE TELL NOW ABOUT U?

Furthermore, it is true that since u≥0:

$$|\widehat{u}(k)| \leqslant \widehat{u}(0) \leqslant \frac{1}{4}$$

First inequality is strict for all k≠0, value signs can be removed for sign ≠

Hence for $k \neq 0$, $\sqrt{1 - 4\widehat{u}(k)} \neq 0$.

Because, $\hat{u}(k) \neq \frac{1}{4}$

$$4\hat{u}(k) \neq 1$$

$$4\hat{u}(k) - 1 \neq 0$$

Square root does not change relation

# THEOREM 1: WHAT DOES THAT SAY ABOUT F?

- Use Riemann-Lebesgue-Theorem:
  - If f is $L^1$integrable on $R^d$ the fouriertransform of f satisfies

$$\hat{f}(z) \equiv \int_{\mathbb{R}^d} f(x) \exp(-iz \cdot x)\, dx \to 0 \text{ as } |z| \to \infty.$$

- It follows that,

At least for large k

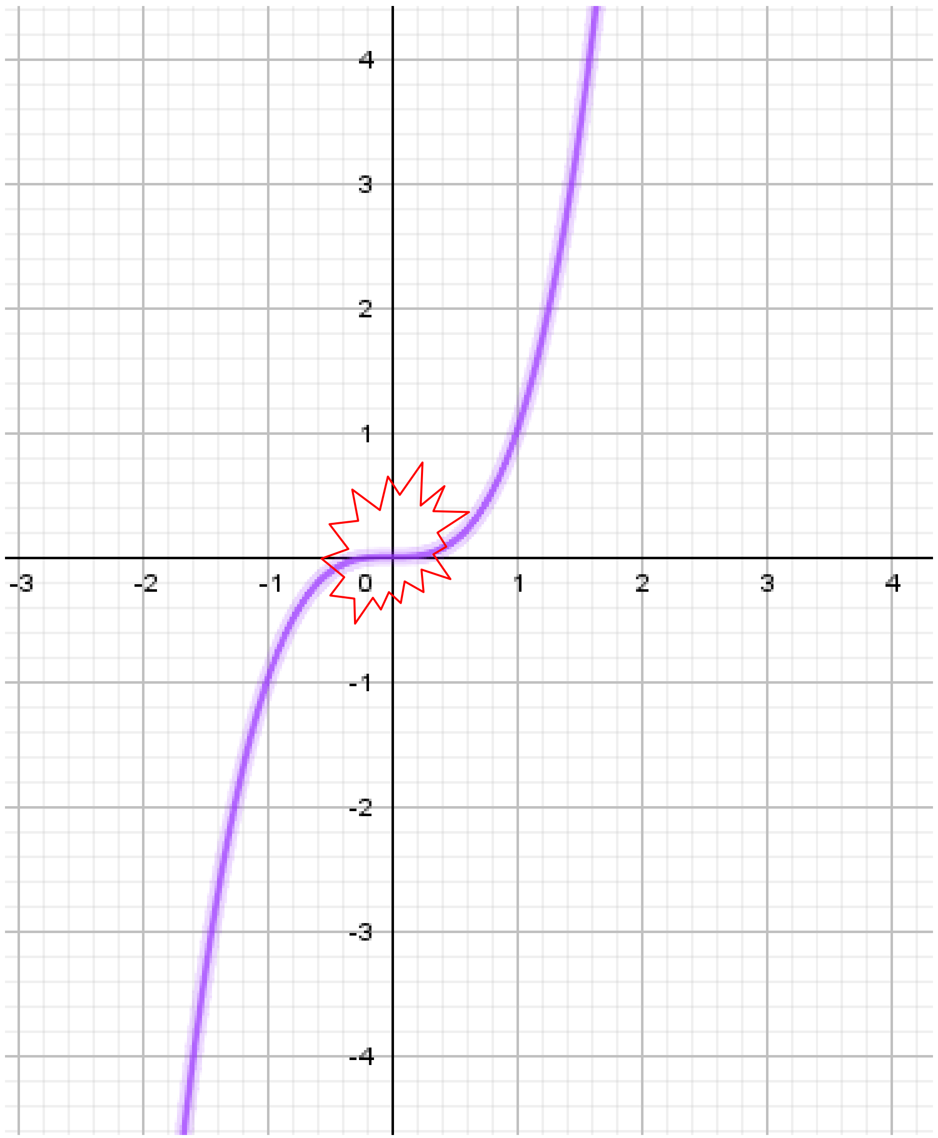
$$\widehat{f}(k) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\widehat{u}(k)}$$

? ✓

$$\left(\hat{f}(k) - \frac{1}{2}\right)^2 = \frac{1}{4} - \hat{u}(k)$$

$$\hat{f}(k) - \frac{1}{2} = -\sqrt{\frac{1}{4}\left(1 - \widehat{U}(k)\right)}$$

$$\hat{f}_2 - \frac{1}{2} = -\frac{1}{2}\sqrt{1 - \hat{u}(k)}$$




Intermediate value theorem, since the function is

$$\widehat{f}(k) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\widehat{u}(k)}$$

# THEOREM 1: PROOF OF SHARP UPPER BOUND

- $\int_{R^d} f(x) \leq 0{,}5$

At $k = 0,\ a = \frac{1}{2} - \sqrt{1-4b}$.

Since u≥0, we know that, the square root is positive, so the inequality is indeed satisfied.

- $\int_{R^d} f(x) \leq 0{,}5$

Remember, how we defined a and b:

- $a \equiv \int_{R^d} f(x)dx \text{ and } b \equiv \int_{R^d} u(x)dx$

Upper bound is sharp, because root can be zero (except for k=0)!

$0 \leq b \leq \frac{1}{4}$

# THEOREM 1: CONVERGENT SERIES

*$f$ is given by the convergent series*

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x)$$

*where the $c_n \geqslant 0$ are the Taylor coefficients in the expansion of $\sqrt{1-x}$*

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n, \quad c_n = \frac{(2n-3)!!}{2^n n!} \sim n^{-3/2}$$

## THEOREM 1: TAKE A SERIES

- Take $c_n = \frac{(2n-3)!!}{2^n n!}$;

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n,$$

A power series:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

- How does that sum look like?

$$1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} + O(x^6)$$

- Apply stirling formula (be careful, no double faculty!)

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \to \infty.$$

→

$$c_n \sim n^{-3/2}$$

## THEOREM 1: CONVERGENCE OF THE SERIES

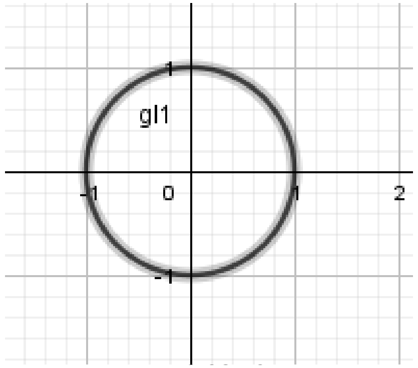
- Now, we got $(1-x)^{\left(\frac{1}{2}\right)} = 1 - \sum_{n=1}^{\infty} w * n^{-\frac{3}{2}} \mathrm{x}^{\mathrm{n}}$
- Does this power series converge?

Yes, it converges absolutely and uniformly on the closed unit disc (convergence radius)

$$\sum_{n=0}^{\infty} |a_n| < \infty$$

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}, so\ that\ \forall n \geq n$$
$$|f_n(x) - f(x)| < \varepsilon.$$

$$\bar{D}_1(P) = \{Q : |P - Q| \leq 1\}.$$



# THEOREM 1: HOW DOES THAT SERIES HELP?

Now, we can try to express the fouriertransformation in terms of this series:

Remember:

$$\frac{1}{4} \geq |\hat{u}(k)|$$

Then, substitute x=$|\hat{u}(k)|$

$$|4\hat{u}(k)| \leqslant 1, \ \sqrt{1 - 4\hat{u}(k)} = 1 - \sum_{n=1}^{\infty} c_n (4\hat{u}(k))^n$$

Element of convergence radius

Careful! Now, cn is in the sum again, therefore the equality is satisfied

# THEOREM 1: HOW CAN WE APPLY THIS TO OUR FUNCTION

- Earlier we got the expression: $\widehat{f}(k) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\widehat{u}(k)}$
- We simply put in our expression for u:

$$\hat{f}(k) = \frac{1}{2} - \frac{1}{2} \cdot \left(1 - \sum_{n=1}^{\infty} c_n\left(\hat{u}(k)\right)^n\right) = 0{,}5 \sum_{n=1}^{\infty} c_n\left(\hat{u}(k)\right)^n$$

# THEOREM 1: FOURIERTRANSFORM BACKWARDS

- Now we can do a „backward fouriertransformation“ to get an expression how a function f, we are looking for looks like!

In general, it is:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{2\pi_i kx}\, dk$$

$$= 0{,}5 \sum_{n=1}^{\infty} c_n \left(\hat{u}(k)\right)^n$$

- Ultimately, we get

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x)$$

Constants, which are independant from k

Remember:

Define $f(x) - f * f(x) \equiv u(x) \geq o$

# THEOREM 1: CONVERGENCE OF F

Does $f(x) = \frac{1}{2}\sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x)$ converge?

We know $\sum_{n=1}^{\infty} c_n$ converges and $\int_{\mathbb{R}^d} 4^n \star^n u(x)\mathrm{d}x \leqslant 1$

Can be treated as a constant

**Also f(x) must converge, since there is no term left that can diverge!**
**F is defined in L^1(R^d)**

# THEOREM 1: POSITIVITY OF F

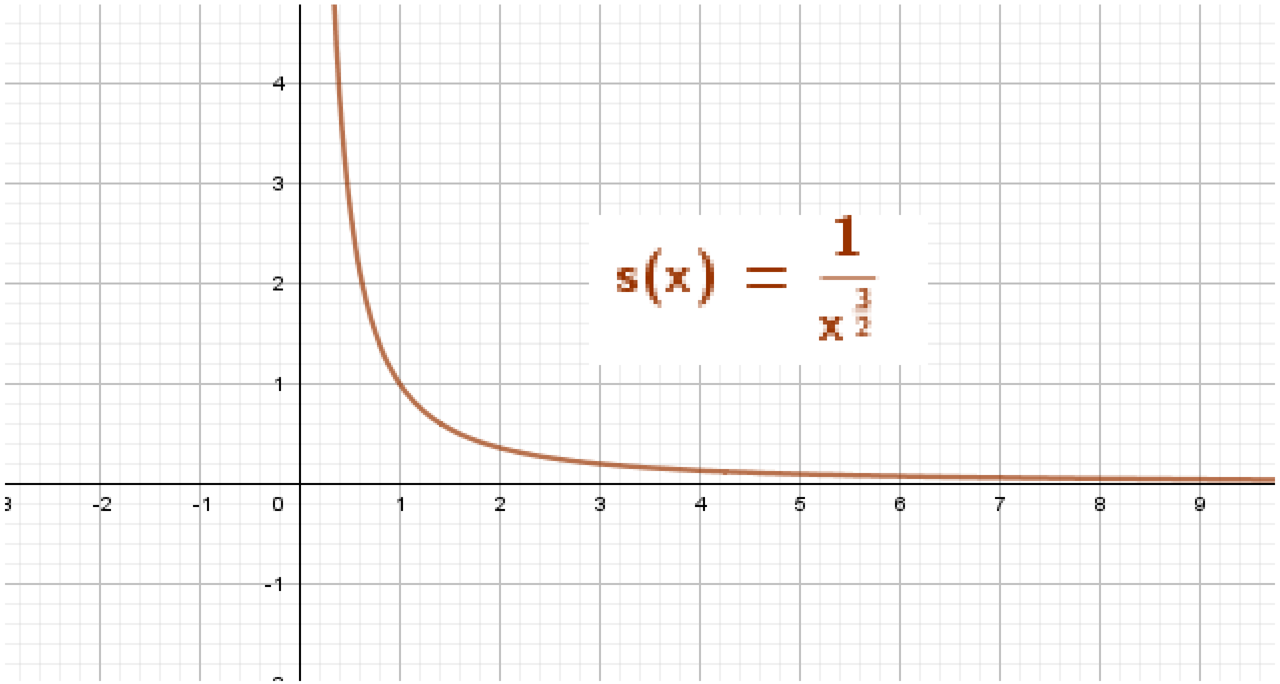
From the definition of the root, it follows that

$$\sum_{n=1}^{\infty} c_n'$$

Must be always positive as well!

4^n is positive as well

U(x) is also positive



# THEOREM 1: CONSEQUENCES OF $U \geq 0$

- If we consider $f(x) = \frac{1}{2}\sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x)$ to be true

We defined that $\widehat{f}(k) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\widehat{u}(k)}$ is true as well

Only equivalences!

- But this is only true if:

$$f(x) - f \star f(x) =: u(x) \geqslant 0$$

f, as defined in the sum, must

$$u(x) \geqslant 0$$

$$\hat{u}(k) \leq \frac{1}{4}$$

**Theorem 1.** *Let $f$ be a real valued function in $L^1(\mathbb{R}^d)$ such that*

$$f(x) - f \star f(x) =: u(x) \geqslant 0 \tag{5}$$

*for all $x$. Then $\int_{\mathbb{R}^d} f(x)\,dx \leqslant \frac{1}{2}$, and $f$ is given by the convergent series*

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*In particular, $f$ is positive. Moreover, if $u \geqslant 0$ is any integrable function with $\int_{\mathbb{R}^d} u(x)\mathrm{d}x \leqslant \frac{1}{4}$, then the sum on the right in (6) defines an integrable function $f$ that satisfies (5).*

We made no further restriction

**Theorem 2.** *Let* $f \in L^1(\mathbb{R}^d)$ *satisfy* (1) *and* $\int_{\mathbb{R}^d} f(x)\, dx = \frac{1}{2}$. *Then* $\int_{\mathbb{R}^d} |x| f(x)\, dx = \infty$.

Slow decay
at infinity

# THEOREM 2: A SPECIAL CASE

- $\int_{R^d} f(x) \leq 0{,}5$

Upper bound is sharp!

$\Rightarrow$ $$\int_{\mathbb{R}^d} 4u(x)\ dx = 1$$

$, a^2 - a = -b,$

b=0,25=0,25-0,5

$=\int_{R^d} u(x)$

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x)$$

$\Rightarrow$ Define: 4u(x)=w(x) as a probability density

$\Rightarrow$ $$f(x) = \sum_{n=0}^{\infty} \star^n w.$$

# THEOREM 2: FINDING AN INEQUALITY INTEGRAL

$$|m| \int_{\mathbb{R}^d} |x| \star^n w(x) \mathrm{d}x \geqslant \int_{\mathbb{R}^d} m \cdot x \star^n w(x) \mathrm{d}x = n|m|^2$$

- How did we get there?
  1. Suppose $|x||f(x)|$ is integrable
  2. Trivial inequality

     $$|m||x| \geqslant m \cdot x$$

  3. Simplify equation

  Why are we doing that?

  We want to show that the first moment can be finite under special conditions

First moments add under convolution [3]

(4)

$$m := \int_{\mathbb{R}^d} x w(x) \mathrm{d}x.$$

[3]Steven W. Smith, in Digital Signal Processing: A Practical Guide for Engineers and Scientists, 2003
(4) "The n-th centered moment of a multiple convolution and its applications to an intercloud gas model "Laury-Micoulaut, C. Astronomy and Astrophysics, vol. 51, no. 3, Sept. 1976, p. 343-346.

$$|m|\int_{\mathbb{R}^d}|x|\star^n w(x)\mathrm{d}x \geqslant \int_{\mathbb{R}^d} m\cdot x\star^n w(x)\mathrm{d}x = n|m|^2$$

## THEOREM 2: WHAT DOES THAT SAY ABOUT F?

$$\int_{\mathbb{R}^d}|x|f(x)\mathrm{d}x \geqslant |m|\sum_{n=1}^{\infty} nc_n = \infty.$$

- Remember, how m was defined

$$m := \int_{\mathbb{R}^d} xw(x)\mathrm{d}x.$$

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} c_n 4^n(\star^n u)(x)$$

→

F(x)= $\sum_{n=0}^{\infty} c_n *^n w$

- $c_n = \frac{(2n-3)!!}{2^n n!} \sim n^{-3/2}$ is never zero → Hence, m must be zero

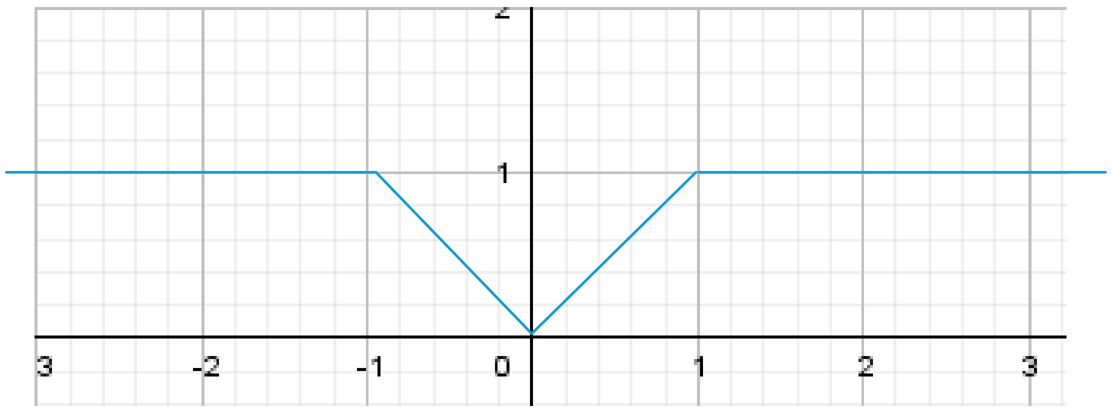
$$\int_{\mathbb{R}^d} |x| \star^n w(x)\mathrm{d}x = \int_{\mathbb{R}^d} |n^{1/2}x| \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x \geqslant n^{1/2} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x.$$

# THEOREM 2: FIND ANOTHER INTEGRAL THAT WILL HELP US

1. Suppose $|x|^2 w(x)$ is integrable, therefore we can find second moment
2. Let us define $\sigma^2$ as the variance of w

$$\sigma^2 = \int_{\mathbb{R}^d} |x|^2 w(x)\mathrm{d}x.$$

3. Define the function $\varphi(x) = \min\{1, |x|\}.$



$$\int_{\mathbb{R}^d} |x| \star^n w(x)\mathrm{d}x = \int_{\mathbb{R}^d} |n^{1/2}x| \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x \geqslant n^{1/2} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x.$$

# THEOREM 2: FIND ANOTHER INTEGRAL THAT WILL HELP US

1. Just as earlier, let´s consider: $\int_{R^d} |x| *^n w(x)dx$

2. Add $n^{0,5}$ in a way, that equality is not lost

3. Make it an inequality
   1. For all n smaller 1 it is:

$$1 < n^{\frac{1}{2}} > n$$

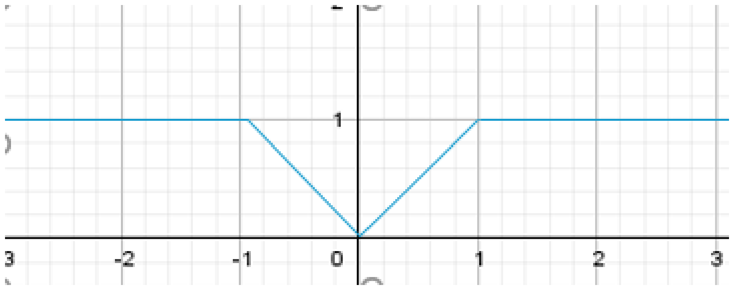
Therefore, $1 < |x| < \left|xn^{\frac{1}{2}}\right|$

   1. For all n larger 1, true as well

Remember:
-moments simply add up under convolution

Remember how we defined phi(x):

$$\varphi(x) = \min\{1, |x|\}.$$



# THEOREM 2: WHAT DOES THAT TELL US ABOUT THE INTEGRAL?

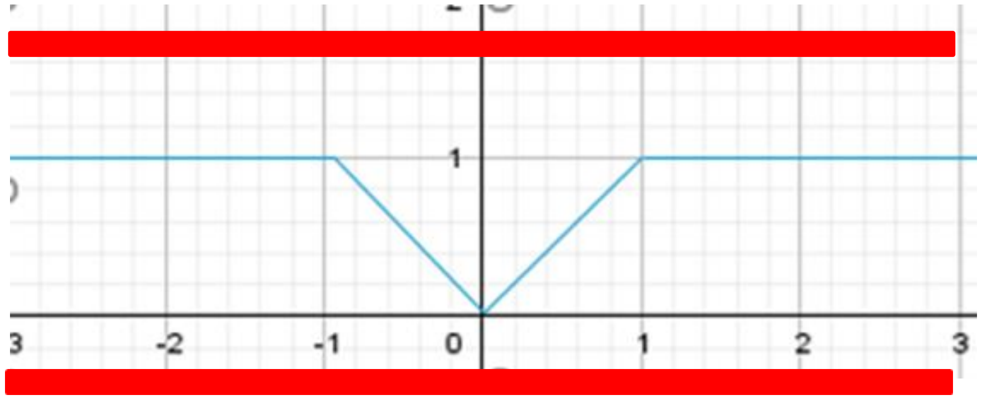
- Use the central limit theorem to find a centered Gaussian probability

define a new probability function $\lim_{n\to\infty} *^n\, w\left(n^{\frac{1}{2}}x\right)n^{\frac{d}{2}} = \gamma(x)$

$$\int_{\mathbb{R}^d} |x| \star^n w(x)\mathrm{d}x = \int_{\mathbb{R}^d} |n^{1/2}x| \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x \geqslant n^{1/2}\int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x.$$

- Phi(x) is bounded

and continuus




$\gamma(x)$

# THEOREM 2: FIND AN UPPER VALUE FOR THE INTEGRAL

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x = \boxed{\int_{\mathbb{R}^d} \varphi(x)\gamma(x)\mathrm{d}x =: C > 0}$$

- Substitute one probabilty function by the CLT with another
- Why is there such a C?
  - Phi(x) is continuous and bounded (at max 1)

$$\int_{-\infty}^{\infty} a\, e^{-(x-b)^2/2c^2}\, dx = \sqrt{2}a\,|c|\,\sqrt{\pi}$$

→ Integral exists

## THEOREM 2: THE FUNCTION DECAYS FAIRLY SLOWLY AT INFINITY

- To proof this, we have to show $\int_{\mathbb{R}^d} |x| f(x)\ dx = \infty.$

- We have already proven: $\int_{R^d} \varphi(x)\gamma(x)dx = C > 0$

- Define C in a new way:

There is a $\delta > 0$ so that for all sufficiently large n

$$\int_{\mathbb{R}^d} |x| \star^n w(x)\mathrm{d}x \geqslant \sqrt{n}\delta,$$

Remember the $\sqrt{n}\ in\ front$

$$n^{1/2} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x.$$

## THEOREM 2: SLOW DECAY: $\int_{\mathbb{R}^d} |x| f(x)\, dx = \infty.$

- Problem: currently we have a definite value for a similar integral

$$\int_{\mathbb{R}^d} |x| \star^n w(x) \mathrm{d}x \geqslant \sqrt{n}\delta,$$

- But now, let´s consider f

$$c_n \sim n^{-\frac{3}{2}}$$

$$\int_{R^d} f(x)|x|dx = \sum_{n=1}^{\infty} c_n \int_{R^d} |x| *^n w(x)\, dx = \infty$$

$$n^{-\frac{3}{2}} * n^{\frac{1}{2}} = \frac{1}{n}$$

→ ***DIVERGES***

To remove the hypothesis that $w$ has finite variance, note that if $w$ is a probability density with zero mean and infinite variance, $\star^n w(n^{1/2}x)n^{d/2}$ is "trying" to converge to a Gaussian of infinite variance. In particular, one would expect that for all $R > 0$,

$$\lim_{n\to\infty} \int_{|x|\leqslant R} \star^n w(n^{1/2}x)n^{d/2}\mathrm{d}x = 0\ , \tag{12}$$

**Theorem 4.** *If $f$ satisfies (5), $\int_{\mathbb{R}^d} xu(x)\mathrm{d}x = 0$ and $\int |x|^2 u(x)\mathrm{d}x < \infty$, then, for all $0 \leqslant p < 1$,*

$$\int |x|^p f(x)\ dx < \infty.$$

# THEOREM 4: WHAT ARE THE NECESSARY REQUIREMENTS?

**Theorem 4.** *If $f$ satisfies (5), $\int_{\mathbb{R}^d} xu(x)\mathrm{d}x = 0$ and $\int |x|^2 u(x)\mathrm{d}x < \infty$, then, for all $0 \leqslant p < 1$,*

$$\int |x|^p f(x)\, dx < \infty.$$

1. Satisfaction of (5) $\quad f(x) - f \star f(x) =: u(x) \geqslant 0$

2. first moment of u is zero

3. Second moment is not infinite

# THEOREM 4: FIND A NEW PROBABILITY FUNCTION

- Exclusion of trivial solution
- Define t: $t = 4\int_{R^d} u(x)dx \leq 1$
- Then, since t>0 we define $w = \frac{4u}{t}$

, $a^2 - a = -b$,

b=0,25=0,25-0,5

$= \int_{R^d} u(x)$

W: $\mathbb{R} \to \mathbb{R}$

1. W is real
2. W is non-negative
3. W is integrable
4. $\int_{R^s} w(x)dx = 1$

→ W is a probabilty density

# THEOREM 4: HOW DOES THAT CORRESPOND TO F?

- We get a new expression for f(x): Use $w=4u/t$ → 4u=t w

$$f(x) = \sum_{n=1}^{\infty} c_n t^n \star^n w(x) \ .$$

Remember, how f was defined:

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x)$$

# THEOREM 4: CARACTERISTICS OF W

W=4u/t

- Mean is zero ⇒ first moment is zero
- Variance $\sigma^2$ is finite ⇒ second moment is finte

Requirements of Theorem 4:

$$\int_{\mathbb{R}^d} xu(x)\mathrm{d}x = 0$$

$$\int |x|^2 u(x)\mathrm{d}x < \infty,$$

# THEOREM 4: HOW DOES THAT HELP WITH F?

Second moments add under convolution

1. Consider the second moment of w(x) convolution:

It is: $\int_{R^d} |x|^2 w(x)dx = \sigma^2$ ⇒ $$\int_{\mathbb{R}^d} |x|^2 \star^n w(x) \mathrm{d}x = n\sigma^2 .$$

2. Use Hölder-inequality for all 0<p<2 $$\int_{\mathbb{R}^d} |x|^p \star^n w(x) \mathrm{d}x \leqslant (n\sigma^2)^{p/2} .$$

Given a measure space and p,q∈ $[0, \infty]$ with $\frac{1}{p} + 1$, Then for all measureable real-oder complex valued functions f and g on the measure space

$$H_p(f) = \left( \int_X |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}}$$

$$H_1(fg) \leq H_p(f) \cdot H_q(g)$$

# THEOREM 4: WHAT CAN WE SAY NOW ABOUT F?

$$\int_{\mathbb{R}^d} |x|^p f(x)\mathrm{d}x \leqslant (\sigma^2)^{p/2} \sum_{n=1}^{\infty} n^{p/2} c_n < \infty$$

- Remember, how f was defined with respect to

$$f(x) = \sum_{n=1}^{\infty} c_n t^n \star^n w(x)$$

$c_n \sim n^{-\frac{3}{2}}$

- We also know from the Hölderinequality:

$$\int_{\mathbb{R}^d} |x|^p \star^n w(x)\mathrm{d}x \leqslant (n\sigma^2)^{p/2}.$$

- Simply put into the equation what we had

$$\int_{R^d} |x|^p f(x)dx = \int_{R^d} \sum_{n=1}^{\infty} c_n \, t^n *^n w(x)\, dx \leq (\sigma^2)^{p/2} \sum_{n=1}^{\infty} n^{p/2} c_n$$

# THEOREM 4: WHY ONLY FOR $0 \leqslant p < 1,$

$$(\sigma^2)^{p/2} \sum_{n=1}^{\infty} n^{p/2} c_n$$

Must converge
Note that, $c_n \sim n^{-\frac{3}{2}}$

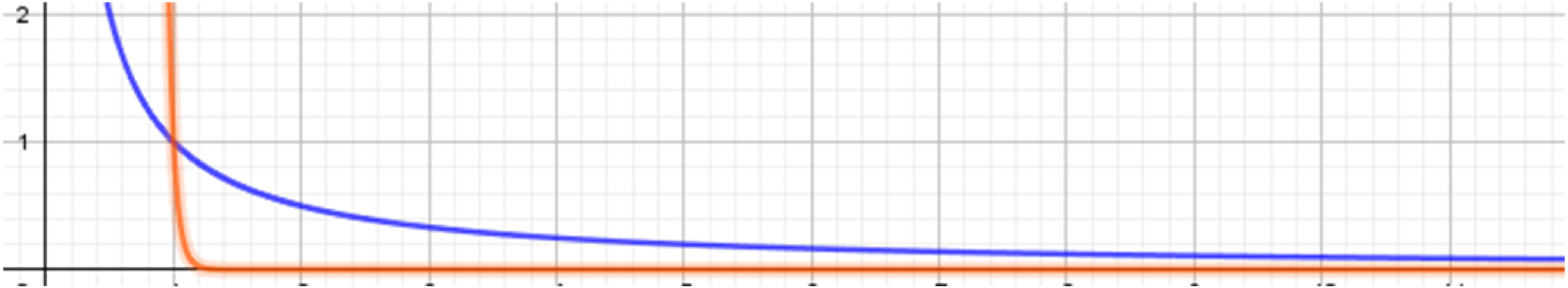
From Theorem 2 we remember: for p=1, this sum diverges

Harmonic sum

But for all smaller p, we can find a majorant sum → sum converges (is smaller than infinity)

→ $$\int |x|^p f(x)\, dx < \infty.$$

# ILLUSTRATION OF (13)



# INTERPRETATION OF THEOREM 2 AND 4

- Theorem 2 implies that wenn the integral is equal to ½ f  cannot decay faster than $|x|^{-(d+1)}$.
- However, integrable solutions f which fufill the convolution inequality and their integral is smaller than ½ can decay quite rapidly, as we saw in illustration (13)

## SUMMARY: WHAT CAN WE SAY ABOUT FUNCTIONS THAT FUFILL $f \geqslant f \star f$

- Are well defined as an element of $L^{p/(2-p)}(\mathbb{R}^d)$ for all $1 \leqslant p \leqslant 2$.
- In $L^1(R^d)$
  - All functions are non-negative
  - The integral of f is smaller or equal to ½
  - ½ is a sharp upper bound
  - If equality is fulfilled, f decays fairly slowly
  - For the inequality f can decay much more rapidly

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