Functional Analysis

Homework Sheet 1

E1.1 [4 Points]**.** Let $(X, \|\cdot\|)$ be a normed vector space, $f : X \longrightarrow \mathbb{R}$ a non-zero linear functional, and $\alpha \in \mathbb{R}$. Prove that the hyperplane $H := \{x \in X : f(x) = \alpha\}$ is closed iff (if and only if) f is continuous.

E1.2 [5 Points]**.** Consider the normed vector space $E := \{u : [0,1] \to \mathbb{R} : u \text{ continuous}, \ u(0) = 0\}$, together with the usual sup-norm $\|u\|_\infty := \sup_{x\in[0,1]} |u(x)|$. Let $T : E \to \mathbb{R}$ be defined by

$$T(u) := \int_0^1 u(t)\, \mathrm{d}t$$

(a) Prove that T defines a continuous linear functional on E and calculate $\|T\|_{E^*} := \sup\limits_{u\in E\setminus\{0\}} \frac{|T(u)|}{\|u\|_\infty}$.

(b) Is the norm of T attained? That is, can we find $u \in E$, s.t. $\|u\|_\infty = 1$ and $T(u) = \|T\|_{E^*}$?

E1.3 [4 Points]**.** Let X be a set and $\bar{\ } : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ a mapping (called a *closure operation*) satisfying

(i) $\bar{\varnothing} = \varnothing$,

(ii) $\forall A \subseteq X : A \subseteq \bar{A} = \bar{\bar{A}}$,

(iii) $\forall A, B \subseteq X$: $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Prove that for any closure operation there exists a unique topology $\mathcal{O}$, s.t. $\forall\ A \subseteq X : \bar{A} = \bar{A}^{\mathcal{O}}$, where $\bar{A}^{\mathcal{O}}$ denotes the closure of A with respect to the topology $\mathcal{O}$.

E1.4 [5 Points]**.** Let X be a normed vector space, $A \subseteq X$ be a closed non-empty set, and let $\phi : X \to \mathbb{R}$ be the distance function

$$\phi(x) := \operatorname{dist}(x, A) := \inf_{a\in A} \|x - a\|\,.$$

Prove:

(a) The map ϕ is Lipschitz-continuous with Lipschitz constant $L = 1$, that is, $\big|\phi(x) - \phi(y)\big| \leqslant \|x - y\|$.

(b) A is convex iff ϕ is convex.

E1.5 [6 Points]**.** Let X be a topological vector space and $A \subseteq X$ be a convex open set with $0 \in A$. Prove that the Minkowski functional

$$p_A(x) := \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in A\right\}$$

satisfies the following properties:

(a) $p_A(\lambda x) = \lambda\, p_A(x)$ for all $\lambda > 0$, $x \in X$,

(b) $p_A(x + y) \leqslant p_A(x) + p_A(y)$ for all $x, y \in X$,

(c) $A = \{x \in X : p_A(x) < 1\}$.

(d) If there exists a norm $\|\cdot\|$ inducing the toplogy of X, the unit ball $B := B_1(0) := \{x \in X : \|x\| < 1\}$ is a convex set and

$$p_B(x) = \|x\| \quad \text{for all } x \in X.$$

Functional Analysis

Homework Sheet 2

E2.1 [5 Points]. Let X be a topological vector space and $(A_i)_{i=1}^N$ a collection of subsets of X. Define the pointwise sum

$$\sum_{i=1}^N A_i := \left\{ \sum_{i=1}^N a_i : a_i \in A_i \right\}.$$

Prove:

(a) If A_i is convex for all $1 \leqslant i \leqslant N$, then $\sum_{i=1}^N A_i$ is convex.

(b) If A_i is open for some $i \in \{1, \dots, N\}$, then $\sum_{i=1}^N A_i$ is open.

E2.2 [7 Points]. Let $(X, \|\cdot\|)$ be a normed vector space. We say that $\|\cdot\|$ is *strictly convex* iff

$$\forall x, y \in X, \text{ s.t. } \|x\| = 1 = \|y\|, x \neq y \; \forall \lambda \in]0,1[: \left\|\lambda x + (1-\lambda)y\right\| < 1.$$

Similarly, a function $\phi : X \longrightarrow]-\infty, \infty]$ is *strictly convex*, iff

$$\forall x, y \in X, \text{ s.t. } x \neq y \; \forall \lambda \in]0,1[: \phi(\lambda x + (1-\lambda)y) < \lambda\phi(x) + (1-\lambda)\phi(y).$$

(a) Prove that $\|\cdot\|$ is strictly convex if and only if $\phi(x) := \|x\|^2$ is strictly convex.

(b) Let $p \in]1, \infty[$. Prove that $\|\cdot\|$ is strictly convex if and only if $\phi(x) := \|x\|^p$ is strictly convex.

E2.3 [5 Points]. Consider the space of real-valued continuous functions $X := C([a,b])$ together with the 2-norm, defined by $\|f\|_2 := \sqrt{\langle f, f\rangle}$ in terms of the scalar product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,\mathrm{d}x, \quad f, g \in X.$$

Prove that $\|\cdot\|_2$ is strictly convex.

E2.4 [7 Points]. Consider the normed spaces of real-valued sequences

$$\ell^\infty := \{(x_n)_{n\in\mathbb{N}} : \sup_{n\in\mathbb{N}} |x_n| < \infty\},$$

$$\ell^1 := \{(x_n)_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |x_n| < \infty\},$$

together with the respective norms $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, and $\|x\|_1 := \sum_{n\in\mathbb{N}} |x_n|$.

(a) Prove that in ℓ^1 infinitely many pairwise disjoint 1-balls $B_{\frac{1}{2}}(y_k) := \{x \in \ell^1 : \|x - y_k\|_1 < \frac{1}{2}\}$, for suitable $y_k \in \ell^1$, $k \in \mathbb{N}$, can be embedded into the unit 1-ball $B_1(0) := \{x \in \ell^1 : \|x\|_1 < 1\}$.

(b) Prove that in ℓ^∞ infinitely many pairwise disjoint ∞-balls of radius $\frac{1}{2}$ can be embedded into the unit ∞-ball $B_1(0) := \{x \in \ell^\infty : \|x\|_\infty < 1\}$.

(c) Prove that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are not equivalent when $\|\cdot\|_\infty$ is restricted to $\ell^1 \subseteq \ell^\infty$.

Functional Analysis

Homework Sheet 3

E3.1 [6 Points]**.** Let (X, d) be a compact metric space. Prove that:

(a) X is complete.

(b) X is bounded, that is, there exist $x \in X$ and $R > 0$ such that $X \subseteq B(x, R)$.

E3.2 [6 Points]**.** Let (X, d) be a complete metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X such that it has no subsequence which is convergent. Prove that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a constant $\varepsilon > 0$ such that $d(x_{n_k}, x_{n_l}) \geqslant \varepsilon \ \forall k \neq l$.

E3.3 [6 Points]**.** Let (X, d) be a complete metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(r_n)_{n\in\mathbb{N}} \subseteq \,]0, \infty[$ such that

$$B(x_1, r_1) \supseteq B(x_2, r_2) \supseteq \dots$$

(a) Prove that if (X, d) is a normed space, (that is, X is a vector space and there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$ for all $x, y \in X$) then

$$\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \neq \varnothing.$$

(b) Find an example of a complete metric space (X, d) and sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(r_n)_{n\in\mathbb{N}} \subseteq \,]0, \infty[$ such that

$$\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} = \varnothing.$$

E3.4 [6 Points]**.** Let (X, d) be a metric space. Prove that there exists a metric d' on X which generates the same topology (that is, $\forall (x_n)_{n\in\mathbb{N}} \subseteq X \ \forall x \in X : x_n \to x$ with respect to d if and only if $x_n \to x$ with respect to d') and satisfies in addition

$$X \subseteq B_{d'}(x, 1) \text{ for all } x \in X.$$

Functional Analysis

Homework Sheet 4

E4.1 [4 Points]**.** Let X be an infinite dimensional normed vector space, $V \subseteq X$ a finite dimensional subspace, and $y \in X \setminus V$. Prove that $\operatorname{dist}(y, V) > 0$ is attained by a vector $v \in V$.

E4.2 [4 Points]**.** Let X be an infinite dimensional normed vector space. Prove that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|x_n - x_m\| \geqslant 1$ for all $n \neq m$.

E4.3 [6 Points]**.** Let $Y := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \ell^1 : \sum_{n=1}^{\infty} \frac{n}{n+1} x_n = 0 \right\}$.

(a) Prove that Y is a closed subspace of ℓ^1.

(b) Let $e_1 := (1, 0, \dots) \in \ell^1$.
Prove or disprove whether $\operatorname{dist}(e_1, Y) := \inf\{\|e_1 - y\|_1 : y \in Y\}$ is attained by a vector $y \in Y$.

(c) Calculate $\operatorname{dist}(e_1, Y)$.

E4.4 [10 Points]**.**

(a) Let Y be a Banach space and $X \subseteq Y$ a subspace. Prove: X closed $\iff$ X Banach.

Prove that the following are Banach spaces:

(b) $C_0(\mathbb{R}^d, \mathbb{C}) := \left\{ f : \mathbb{R}^d \to \mathbb{C} : f \text{ continuous}, \lim\limits_{|x|\to\infty} |f(x)| = 0 \right\}$ with norm $\|f\| := \sup\limits_{x\in\mathbb{R}^d} |f(x)|$.

(c) $c_0(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{C}, \lim\limits_{n\to\infty} |x_n| = 0 \right\}$ with $\|x\| := \sup\limits_{n\in\mathbb{N}} |x_n|$.

(d) $X := \left\{ f : [0,1] \to \mathbb{C} : f \text{ continuously differentiable and } f(0) = 0 \right\}$ with $\|f\| := \sup\limits_{x\in[0,1]} |f'(x)|$.

(e) Prove that $X := \{f : [0,1] \to \mathbb{C} : f \text{ continuous}\}$ with

$$\|f\| := \int_0^1 x^2 |f(x)| \, \mathrm{d}x$$

is a normed space which is not complete.

Functional Analysis

Homework Sheet 5

E5.1 [6 Points]**.** Let X be a Banach space, Y normed, and $(T_i)_{\iota \in I} \subseteq \mathcal{L}(X,Y)$ an arbitrary collection of bounded linear operators such that $\sup\{\|T_\iota\| : \iota \in I\} = \infty$.

Prove directly (without using the Baire or Uniform Boundedness Theorem) that there exists an $x \in X$ such that $\sup\{\|T_\iota x\| : \iota \in I\} = \infty$.

Remark and Hints: The goal of this problem is to obtain an elementary proof of the Uniform Boundedness Principle, which does not use the Baire Theorem. We suggest to use the following strategy:

(a) Show that there exist sequences $(T_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that:

$$\|T_n x_n\| \geqslant n + \sum_{j=1}^{n-1} \|T_n x_j\|, \text{ for } n \geqslant 2, \quad \text{and } \|T_1 x_1\| \geqslant 1,$$
$$\|x_n\| \leqslant 2^{-n} \min\{\|T_j\|^{-1} : j < n\}, \text{ for } n \geqslant 2, \quad \text{and } \|x_1\| \leqslant 1/2.$$

(b) Show that the series $\sum_{n\in\mathbb{N}} x_n$ converges to some $x \in X$.

(c) Show that $\sum_{j=n+1}^{\infty} \|T_n x_j\| \leqslant 1$ for each n.

(d) Show that $\|T_n x\| \geqslant n - 1$ for each n, so $\sup\{\|T_i x\| : i \in I\} = +\infty$.

E5.2 [4 Points]**.** Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$ be an isomorphism.
Prove: $\overline{T(B_X(0,1))} = T\Big(\overline{B_X(0,1)}\Big)$.

E5.3 [4 Points]**.** Let X, Y be Banach spaces and $T : X \to Y$ be linear s.t. $\overline{T(B_X(0,1))} = \overline{B_Y(0,1)}$.
Prove: $T(B_X(0,1)) = B_Y(0,1)$.

E5.4 [5 Points]**.** Let X be a normed space and $M \subseteq X$ a closed subspace. Recall that the quotient space $X/M := \{[x]_\sim = x + M : x \in X\}$ consists of all equivalence classes of the relation $x \sim y :\Leftrightarrow x - y \in M$.
Prove that $\|[x]\| := \operatorname{dist}(x, M)$ defines a norm on X/M.

E5.5 [5 Points]**.** Let X be a separable normed space. Prove that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq X^*$ with $\|f_n\| = 1$ for all $n \in \mathbb{N}$, such that for all $x \in X$ we have $\|x\| = \sup_{n\in\mathbb{N}} |f_n(x)|$.

Functional Analysis

Homework Sheet 6

E6.1 [6 Points]**.** Let X be a Banach space and let $M \subseteq X$ be a closed subspace with finite codimension, that is,

$$\operatorname{codim} M := \dim(X/M) < \infty.$$

Prove that M is complemented in X.

E6.2 [6 Points]**.** Let X be a Banach space, $M \subseteq X$ a closed subspace of X, and $q : X \to X/M$ the quotient map. Prove that

$$q(B_X(0,1)) = B_{X/M}(0,1).$$

E6.3 [6 Points]**.** Let X, Y be Banach spaces, $f \in \mathcal{L}(X,Y)$. Prove that the following statements are equivalent:

(i) $f(X)$ is closed in Y.

(ii) $X/\ker f$ is isomorphic to $f(X)$.

E6.4 [6 Points]**.** Let X be an infinite-dimensional Banach space.

Prove that the weak topology $\sigma(X, X^*)$ is not metrizable.

Hint: To argue by contradiction, suppose that there exists a metric $d(x,y)$ on X such that d induces the topology $\sigma(X, X^*)$. You may then use the following strategy:

(a) For every $n \in \mathbb{N}$, let $V_n \subseteq X$ be a neighbourhood of zero in $\sigma(X, X^*)$ such that

$$V_n \subseteq \{x \in X : d(x, 0) < 1/n\}.$$

Prove that there exists a sequence $(f_k)_{k\in\mathbb{N}} \subseteq X^*$ such that every $g \in X^*$ is a finite linear combination of the f_k.

(b) Deduce from (a) that X^* is finite-dimensional (e.g. using the Baire theorem).

(c) Prove that $\sigma(X, X^*)$ is not metrizable.

Functional Analysis

Homework Sheet 7

E7.1 [8 Points]**.** Let X be a Banach space and denote by T the canonical embedding of X into X^{**}, defined by

$$T : \begin{cases} X \longrightarrow X^{**} \\ x \longmapsto T_x \end{cases} \quad \text{with} \quad T_x : \begin{cases} X^* \longrightarrow \mathbb{K} \\ f \longmapsto T_x(f) := f(x) \end{cases}$$

Prove:

(a) $\|T_x\|_{X^{**}} = \|x\|_X$

(b) $x_n \rightharpoonup x$ weakly in X if and only if $T_{x_n} \overset{*}{\rightharpoonup} T_x$ weakly-$*$ in X^{**}

(c) If $x_n \rightharpoonup x$ weakly in X, then the set $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

E7.2 [10 Points]**.** Let X be a Banach space with $\dim X = \infty$.
Prove:

(a) There exists a non-empty subset $A \subseteq X$ such that:

- $0 \notin A$ and $A \cap B(0,r)$ is finite for all $r > 0$.
- For every finite family $\{f_i\}_{i\in I} \subseteq X^*$ we have

$$\inf_{x\in A} \sup_{i\in I} |f_i(x)| = 0.$$

(b) The set A is not closed in the weak topology $\sigma(X, X^*)$.

(c) For every sequence $\{y_n\} \subseteq A$, if $y_n \rightharpoonup y$ weakly, then $y \in A$.
(That is, A is weakly sequentially closed.)

(d) Prove using (a)–(c) that $(X, \sigma(X, X^*))$ is not metrizable.

E7.3 [6 Points]**.** Let X be a reflexive Banach space. Prove:

(a) For all $f \in X^*$ there exists an $x_f \in X$ with $\|x_f\| = 1$ such that $\|f\| = f(x_f)$.

(b) If M is a closed subspace of X and $x_0 \notin M$ then there exists an $x_1 \in M$ such that

$$\|x_0 - x_1\| = \inf_{x\in M} \|x_0 - x\|$$

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Functional Analysis

Homework Sheet 8

E8.1. [2 points] Let X be a Banach space. Prove that the ball $B = \overline{B(0,1)}^{\|\cdot\|_{X^*}}$ is closed in X^* with the weak-* topology.

E8.2. [4 points] Let X be a Banach space. Prove that the following statements are equivalent:

(i) X is uniformly convex.

(ii) For all sequences $\{x_n\}$, $\{y_n\}$, if $\|x_n\| \le 1, \|y_n\| \le 1$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

E8.3. [4 points] Let X be a uniformly convex Banach space and let $\{x_n\}_{n=1}^\infty \subset X$. Prove that if $x_n \rightharpoonup x$ weakly and $\|x_n\| \to \|x\|$, then $x_n \to x$ strongly.

E8.4. [4 points] Let X be a Banach space. Let $T : X \to X^{**}$ be the canonical mapping

$$T_x(f) = f(x), \quad \forall f \in X^*, x \in X.$$

Let S and S_{**} be the unit spheres in X and X^{**}. Prove that $T(S)$ is dense in S_{**}. (Note: Goldstine Lemma says that $T(B)$ is dense in B_{**} with closed balls.)

E8.5. [4 points] Let X be a reflexive Banach space.

(a) Prove that if M is a closed subspace of X, then M is reflexive.

(b) Prove that if Y is a Banach space isomorphic to X, then Y is reflexive.

E8.6. [6 points] Let X be a Banach space and let M be a subspace of X. Let $q : X \to X/M$ be the quotient map.

(a) Prove that U is weakly open in X/M if and only if $q^{-1}(U)$ is weakly open in X.

(b) Prove that if X is reflexive, then X/M is reflexive.

(c) Prove that if M and X/M are reflexive, then X is reflexive.

Lösungen können bis zum 18.06.2021 um 14:00 Uhr über Uni2Work abgegeben werden.
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Functional Analysis

Homework Sheet 9

E9.1. [4 points] Let (Ω, μ) be a measure space with $\mu(\Omega) < \infty$.

(a) Prove that $L^p(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq p \leq \infty$.

(b) Prove that for every $f \in L^\infty(\Omega)$ we have

$$\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty}.$$

E9.2. [6 points] Consider $\mathbb{R}^d$ with the Lebesgue measure.

(a) Prove that for $1 \leq p < q \leq \infty$, both $L^p(\mathbb{R}^d) \backslash L^q(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d) \backslash L^p(\mathbb{R}^d)$ are non-empty.

(b) Prove that for $1 \leq p \leq \infty$, there exists a function $f \in L^p(\mathbb{R}^d)$ such that f does not belong to $L^q(\mathbb{R}^d)$ for all $1 \leq q \leq \infty$, $q \neq p$.

E9.3. [4 points] (a) Let (Ω, μ) be a measure space and $1 \leq p \leq q \leq r \leq \infty$. Prove that

$$L^q(\Omega) \subset L^p(\Omega) + L^r(\Omega) = \{f + g, f \in L^p(\Omega), g \in L^r(\Omega)\}.$$

(b) Consider $\mathbb{R}^d$ with $d \geq 2$ and the Lebesgue measure. Prove that $f(x) = |x|^{-1}$ does not belong to L^q space for any $1 \leq q \leq \infty$, but it belongs to $L^p + L^r$ for some $1 < p < r < \infty$.

E9.4. [4 points] Let (Ω, μ) be a measure space and let $p, q \in (1, \infty)$. Let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in $L^q(\Omega)$.

(a) Prove that if $f_n \to f$ strongly in $L^p(\Omega)$, then $f_n \to f$ strongly in $L^r(\Omega)$ for all r between p and q, $r \neq q$.

(b) Prove that if $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$, then $f_n \rightharpoonup f$ weakly in $L^r(\Omega)$ for all r between p and q, including $r = q$.

E9.5. [6 points] Let (Ω, μ) be a measure space which is sigma-finite. Let $1 < p < \infty$. Let $f : \Omega \to \mathbb{C}$ be a measurable function such that $fg \in L^1(\Omega)$ for all $g \in L^p(\Omega)$. Prove that $f \in L^q(\Omega)$ with $1/p + 1/q = 1$.

Lösungen können bis zum 25.06.2021 um 14:00 Uhr über Uni2Work abgegeben werden.
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Functional Analysis

Homework Sheet 10

E10.1. [4 points] Let $d \geq 1$. Find a sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $f_n \to 0$ strongly in $L^1(\mathbb{R}^d)$, $f_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$, and f_n does not converge to 0 strongly in $L^2(\mathbb{R}^d)$.

E10.2. [4 points] Let $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$. For every $r > 0$, define

$$f_r(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \mathrm{d}y.$$

Prove that $f_r \in L^p(\mathbb{R}^d)$ for all $r > 0$, and $f_r \to f$ strongly in $L^p(\mathbb{R}^d)$ when $r \to 0$.

E10.3. [4 points] Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^d} f(x)\varphi(x) \mathrm{d}x \geq 0, \quad \forall 0 \leq \varphi \in C_c^{\infty}(\mathbb{R}^d).$$

Prove that $f \geq 0$.

E10.4. [4 points] Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Prove that if $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g \in C_0(\mathbb{R}^d)$, namely $x \mapsto (f * g)(x)$ is continuous and $\lim_{|x| \to \infty} (f * g)(x) = 0$.

E10.5. [4 points] Prove that if $f \in L^1(\mathbb{R}^d)$, then $\widehat{f} \in C_0(\mathbb{R}^d)$, namely $k \mapsto \widehat{f}(k)$ is continuous and $\lim_{|k| \to \infty} \widehat{f}(k) = 0$.

E10.6. [4 points] Let $f(x) = e^{-\pi|x|^2}$ with $x \in \mathbb{R}^d$. Prove that

$$\widehat{f}(k) = e^{-\pi|k|^2}, \quad \forall k \in \mathbb{R}^d.$$

Lösungen können bis zum 02.07.2021 um 14:00 Uhr über Uni2Work abgegeben werden.
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Functional Analysis

Homework Sheet 11

E11.1. [8 points] (a) Prove that $(\ell^1(\mathbb{N}))^* = \ell^\infty(\mathbb{N})$.

(b) Prove that $(\ell^\infty(\mathbb{N}))^*$ is really bigger than $\ell^1(\mathbb{N})$ and that $\ell^\infty(\mathbb{N})$ is not separable.

E11.2. [8 points] Let X be a vector space. Let $T : X \times X \to \mathbb{C}$ be a function satisfying that $T(x, y)$ is linear in y and anti-linear in x.

(a) Prove that if we assume further that $T(x, x) \in \mathbb{R}$ for all $x \in X$, then

$$\overline{T(x, y)} = T(y, x), \quad \forall x, y \in X.$$

(b) Prove that if we assume further that $T(x, x) \geq 0$ for all $x \in X$, then

$$|T(x, y)|^2 \leq T(x, x)T(y, y), \quad \forall x, y \in X.$$

E11.3. [4 points] Let X be a Hilbert space. Let M, N be two closed subspaces of X such that $M \perp N$, namely $\langle x, y \rangle = 0$ for all $x \in M$ and $y \in N$. Prove that $M + N$ is a closed subspace of X.

E11.4. [4 points] Let X be a Hilbert space. Let $\{u_n\}_{n=1}^\infty \subset X$ such that $u_n \rightharpoonup 0$ weakly. Prove that there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ such that

$$v_m := \frac{1}{m} \sum_{k=1}^{m} u_{n_k}$$

converges strongly to 0 when $m \to \infty$.

Lösungen können bis zum 09.07.2021 um 14:00 Uhr über Uni2Work abgegeben werden.
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Functional Analysis

Homework Sheet 12

E12.1. [2 points] Let X be a Hilbert space and let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal family in X. Prove that $x_n \rightharpoonup 0$ weakly in X.

E12.2. [8 points] Let X be a Hilbert space and let $B \in \mathcal{L}(X)$ such that $\|B\| < 1$. Define

$$S_n = 1 + B + B^2 + ... + B^n.$$

(a) Prove that S_n is a Cauchy sequence in $\mathcal{L}(X)$.

(b) Prove that $S_n(1 - B) = (1 - B)S_n \to 1$ strongly in $\mathcal{L}(X)$ when $n \to \infty$.

(c) Deduce that $(1 - B)^{-1} = S_\infty$ where $S_\infty = \lim_{n\to\infty} S_n = 1 + B + B^2 + ...$

E12.3. [8 points] Let (Ω, μ) be a measure space which is sigma-finite. Let $X = L^2(\Omega, \mu)$. Let $a \in L^\infty(\Omega, \mu)$. Let A be an operator on X defined by

$$(Af)(x) = a(x)f(x), \quad \forall f \in X, x \in \Omega.$$

(a) Prove that $A \in \mathcal{L}(X)$ and $\|A\| = \|a\|_{L^\infty}$.

(b) Prove that $\sigma(A) = \text{ess-range}(a)$ where

$$\text{ess-range}(a) = \{\lambda \in \mathbb{C} : \mu(\{x \in \Omega : |a(x) - \lambda| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0\}.$$

E12.4. [6 points] Let $G \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let A be an operator on $L^2(\mathbb{R}^d)$ defined by

$$(Af)(x) = (G * f)(x) = \int_{\mathbb{R}^d} G(x - y)f(y)\mathrm{d}y, \quad \forall f \in L^2(\mathbb{R}^d).$$

(a) Prove that A is bounded and $\|A\| \le \|G\|_{L^1}$.

(b) Let $\mathcal{F}$ be the Fourier transform. Prove that $\mathcal{F}A\mathcal{F}^{-1}$ is the multiplication operator by $\widehat{G}$ in $L^2(\mathbb{R}^d)$, namely

$$(\mathcal{F}A\mathcal{F}^{-1}g)(k) = \widehat{G}(k)g(k), \quad \forall g \in L^2(\mathbb{R}^d).$$

Lösungen können bis zum 16.07.2021 um 14:00 Uhr über Uni2Work abgegeben werden.
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