Large components of random geometric graphs
Mini-course: Munich "Summer" School 2022
on Discrete Random Systems

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1 Introduction and Preliminaries

1.1 Overview

Given $d \in \mathbb{N}$ and finite $\mathcal{X} \subset \mathbb{R}^d$, and $r > 0$, the **geometric graph** $G(\mathcal{X}, r)$ has vertex set $\mathcal{X}$ and edge set $\{\{x, y\} : \|x - y\| \leq r\}$, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$.

Motivation: wireless communications. Spatial epidemics. Topological data analysis.

For a **random** geometric graph (RGG):
take $\mathcal{X}$ to be a random set of points.
Let $\xi_1, \xi_2, \ldots$ be independent random d-vectors, uniformly distributed over the set $B(1) := [-1/2, 1/2]^d$ (a box of side 1). Set

$$\mathcal{X}_n := \{\xi_1, \ldots, \xi_n\}.$$

One reason to study RGGs is to explore 'typical' properties of geometric graphs. Another reason is to assess statistical tests (e.g. for unifomity) based on the graph $G(\mathcal{X}_n, r_n)$.

Book [3]: *Random Geometric Graphs* (2003).
In this course we consider the RGG $G(\mathcal{X}_n, r_n)$ with $(r_n)_{n \geq 1}$ a specified sequence of distance parameters. For simplicity we assume from now on that $d = 2$, although many of the ideas here can be extended to higher dimensions.

Notation: for asymptotics as $n \to \infty$.
For $(0, \infty)$-valued sequences a_n and b_n:

- $a_n = O(b_n)$ means $\limsup(a_n/b_n) < \infty$.
- $a_n = \Theta(b_n)$ means that both $a_n = O(b_n)$ and $b_n = O(a_n)$.
- $a_n \sim b_n$ means $a_n/b_n \to 1$.

- 'With high probability' or 'w.h.p.' means 'with probability tending to 1 as $n \to \infty$'.

(All our asymptotics are as $n \to \infty$.)
We investigate the following questions for $G(\mathcal{X}_n, r_n)$, asymptotically as $n \to \infty$:

- How large does r_n have to be for $G(\mathcal{X}_n, r_n)$ to be **connected** w.h.p.?
- How large does r_n have to be for $G(\mathcal{X}_n, r_n)$ to have a **giant component** containing a non-vanishing proportion of the vertices, w.h.p.?

One interpretation of the RGG is as a (crude) model of a spatial epidemic, starting from a single infected individual. The first question above relates to whether the entire population becomes infected; the second question relates to whether a significant proportion of the population become infected.

Similar questions have been studied for (e.g.) the **Erdös-Rényi random graph** $G(n, p)$, defined as follows. There are n vertices, and for each pair of vertices, an edge between them is included with probability p, independently of the other pairs. However, different methods are needed for RGGs.

First question: **connectivity** of $G(n, r_n)$.

Let $N_0(n)$ be the number of isolated vertices.

If $n\pi r_n^2 \sim \alpha \log n$ as $n \to \infty$, then

$$\begin{aligned}\mathbb{E}[N_0(n)] &= n\mathbb{P}[\text{Degree}(\xi_1) = 0] \\ &\approx n(1 - \pi r_n^2)^{n-1} \\ &\approx n \exp(-(n-1)\pi r_n^2) \approx n^{1-\alpha},\end{aligned}$$

which suggests the following:

Ex. 1.1. Suppose $n\pi r_n^2 / \log n \to \alpha$. Prove
$\mathbb{E}[N_0(n)] \to \infty$ if $\alpha < 1$,
$\mathbb{E}[N_0(n)] \to 0$ if $\alpha > 1$.

This suggests that if $\alpha < 1$ then $G(\mathcal{X}_n, r_n)$ is unlikely to be connected (because there are lots of isolated vertices).

In fact, if $\alpha > 1$ then $G(\mathcal{X}_n, r_n)$ is likely to be connected, answering our first question:

Theorem 1.2. *If $n\pi r_n^2 / \log n \to \alpha \in (0, \infty)$:*
if $\alpha > 1$ then $G(\mathcal{X}_n, r_n) \in \mathcal{K}$ w.h.p., but
if $\alpha < 1$ then $G(\mathcal{X}_n, r_n) \notin \mathcal{K}$ w.h.p.

Here $\mathcal{K}$ denotes the class of connected graphs.

2nd question: **Giant component** of $G(\mathcal{X}_n, r_n)$.

Consider the degree of a 'typical' vertex:

Exercise 1.3. Prove that if $r_n \to 0$, then $\mathbb{E}[\text{Degree}(\xi_1)] \sim n\pi r_n^2$.

Thus if we take the **thermodynamic limit** with $nr_n^2 \to \lambda$ for some $\lambda \in (0, \infty)$, then the 'average degree' approximates to $\pi\lambda$.

It turns out that if λ exceeds a certain critical value λ_c, then $G(n, r_n)$ has a giant component w.h.p.:

Theorem 1.4. *If $nr_n^2 \to \lambda \in (0, \infty)$ as $n \to \infty$, then the order of the largest component of $G(\mathcal{X}_n, r_n)$, divided by n, converges in probability to a limit $p_\infty(\lambda)$. There is a critical value $\lambda_c \in (0, \infty)$ such that $p_\infty(\lambda) = 0$ for $\lambda \leq \lambda_c$ and $p_\infty(\lambda) > 0$ for $\lambda > \lambda_c$.*

The $p_\infty(\lambda)$ appearing in this result is a continuum percolation function, that we shall discuss in more detail later on.

Theorems 1.2 and 1.4 are proved in [3]. In this course we shall prove 'Poissonized' versions.

1.2 Poissonization

A set $\mathcal{X} \subset \mathbb{R}^2$ is said to be *locally finite* if $\mathcal{X}(B) < \infty$ for all bounded $B \subset \mathbb{R}^2$. Here $\mathcal{X}(B)$ means the number of points of $\mathcal{X}$ in B.

For bounded measurable $g : \mathbb{R}^2 \to [0, \infty)$, a *Poisson process* in $\mathbb{R}^2$ with *intensity function* g is a random, locally finite subset $\mathcal{P}$ of $\mathbb{R}^2$ such that for all disjoint Borel $A_1, \ldots, A_k \subset S$;

- $\mathcal{P}(A_1) \sim \text{Poisson}\left(\int_{A_1} g(x)dx\right)$
- $\mathcal{P}(A_1), \ldots, \mathcal{P}(A_k)$ are independent.

In the special case where $g = \lambda \mathbf{1}_S$, for some constant $\lambda > 0$ and some Borel set $S \subseteq \mathbb{R}^2$, we refer to $\mathcal{P}$ as a **homogeneous Poisson process** in S with **intensity** λ.

Book [2]: *Lectures on the Poisson Process* (2018).

For $s, \lambda > 0$, set $B(s) := [-s/2, s/2]^2$.

We can generate a homogeneous Poisson process $\mathcal{H}_{\lambda,s}$ of intensity λ on $B(s)$ as follows.

Let $\xi_1, \xi_2, \cdots$ be independent and uniform over $B(1)$.

Let $N_{\lambda s^2}$ be Poisson distributed with parameter λs^2, independent of $(\xi_1, \xi_2, \ldots)$, and set

$$\mathcal{H}_{\lambda,s} := \{s\xi_1, \ldots, s\xi_{N_{\lambda s^2}}\}. \tag{1.1}$$

Exercise 1.5. Prove that $\mathcal{H}_{\lambda,s}$ is a homogeneous Poisson process in $B(s)$ with intensity λ.

We also write $\mathcal{P}_n$ for $\mathcal{H}_{n,1}$. The 'Poissonized' version of the RGG $G(\mathcal{X}_n, r_n)$ is the graph $G(\mathcal{P}_n, r_n)$.

We now state some basic facts about Poisson processes (see [2] for proofs).

Theorem 1.6. (Superposition) *Suppose $\mathcal{P}$, $\mathcal{P}'$ are independent Poisson process in $\mathbb{R}^2$ with intensity functions $g(\cdot)$ and $g'(\cdot)$ respectively. Then $\mathcal{P} \cup \mathcal{P}'$ is a Poisson process in $\mathbb{R}^2$ with intensity function $g(\cdot) + g'(\cdot)$.*

Theorem 1.7. (Thinning) *Suppose $\mathcal{P}$ is a Poisson process in $\mathbb{R}^2$ with intensity function $g(\cdot)$ and $0 < p < 1$. Let each point of $\mathcal{P}$ be* **accepted** *with probability p, independently of all other points; let $\mathcal{P}'$ be the point process of accepted points. Then $\mathcal{P}'$ is a Poisson process in $\mathbb{R}^2$ with intensity function $pg(\cdot)$.*

Theorem 1.8. (Mecke formula.) *Let $k \in \mathbb{N}$. Let $\lambda, s > 0$. For any measurable real-valued function f, defined on the product of $(\mathbb{R}^2)^k$ and the space of finite subsets of $B(s)$, for which it exists, the expectation of*

$$\sum_{x_1,\ldots,x_k \in \mathcal{H}_{\lambda,s}}^{\neq} f(x_1,\ldots,x_k,\mathcal{H}_{\lambda,s} \setminus \{x_1,\ldots,x_k\})$$

equals

$$\lambda^k \int_{B(s)} dx_1 \cdots \int_{B(s)} dx_k \mathbb{E} f(x_1,\ldots,x_k,\mathcal{H}_{\lambda,s})$$

where $\sum^{\neq}$ means the sum is over ordered k-tuples of distinct points of $\mathcal{H}_{\lambda,s}$.

For $\lambda > 0$, let $\mathcal{H}_\lambda$ be a homogeneous Poisson process of intensity λ in the whole of $\mathbb{R}^2$. Given $\mathcal{X} \subset \mathbb{R}^2$ and $y \in \mathbb{R}^2$, write $\mathcal{X} + y$ for $\{x + y : x \in \mathcal{X}\}$.

Theorem 1.9 (Translation and rotation invariance)**.** *Let $\lambda > 0$, $y \in \mathbb{R}^d$, and let ρ be any rotation of $\mathbb{R}^2$. Then the point process $\mathcal{H}_\lambda + y$ is also a homogeneous Poisson process in $\mathbb{R}^2$ of intensity λ, as is $\rho(\mathcal{H}_\lambda)$.*

Let $\mathbf{o} := (0, 0)$, the origin in $\mathbb{R}^2$.

Theorem 1.10. (Mecke formula for infinite Poisson process) *Suppose $h(x;\mathcal{X})$ is a bounded measurable real-valued function defined on all pairs of the form $(x,\mathcal{X})$ with $\mathcal{X}$ a locally finite subset of $\mathbb{R}^2$. Assume that h is translation-invariant, meaning that $h(x;\mathcal{X}) = h(\mathbf{o};\mathcal{X} + (-x))$ for any $(x,\mathcal{X})$. Then*

$$\mathbb{E} \sum_{x \in \mathcal{H}_\lambda \cap B(s)} h(x;\mathcal{H}_\lambda \setminus \{x\}) = \lambda s^2 \mathbb{E}\left[h(\mathbf{o};\mathcal{H}_\lambda)\right].$$

Proof. (sketch) LHS equals $\lambda \int_{B(s)} \mathbb{E}[h(x,\mathcal{H}_\lambda)]$ by Mecke. By translation invariance, this equals the RHS. □

2 Connectivity

Recall $\mathcal{K} := \{\text{connected graphs}\}$. Let

$$\rho_n = \min\{r : G(\mathcal{P}_n, r) \in \mathcal{K}\}$$

which is a random variable, called the *connectivity threshold*. In this section we prove:

Theorem 2.1. *It is the case that*

$$n\pi\rho_n^2/\log n \xrightarrow{P} 1. \tag{2.1}$$

Given r_n, let δ_n denote the minimum degree of $G(\mathcal{P}_n, r_n)$.
For $x \in \mathbb{R}^2$ and $r > 0$ define the disk

$$D(x,r) := \{y \in \mathbb{R}^2 : \|y - x\| \le r\}.$$

Also let $\text{Leb}(\cdot)$ denote area (2-dimensional Lebesgue measure).

Theorem 2.2. *If $n\pi r_n^2/\log n = \alpha < 1$ for all $n \geq 2$ then $\mathbb{P}[\delta_n = 0] \to 1$.*

Proof. Let N_n here denote the number of vertices of $\mathcal{P}_n \cap B(1/2)$ of degree zero in $G(\mathcal{P}_n, r_n)$ (we restrict to the smaller square $B(1/2)$ to avoid boundary effects). Then by the Mecke formula,

$$\begin{aligned} \mathbb{E}[N_n] &= n \int_{B(1/2)} \mathbb{P}[\mathcal{P}_n(D(x, r_n)) = 0] dx \\ &= (n/4) \exp(-n\pi r_n^2) = (1/4) n^{1-\alpha}, \end{aligned}$$

which tends to infinity as $n \to \infty$.

Also $N_n(N_n - 1)$ is the number of ordered pairs (x, y) of distinct points in $\mathcal{P}_n \cap B(1/2)$ such that $(\mathcal{P}_n \setminus \{x, y\})(D(x, r_n) \cup D(y, r_n)) = 0$ and $\|y - x\| > r_n$. By the Mecke formula $\mathbb{E}[N_n(N_n - 1)]$ equals

$$n^2 \int_{B(1/2)} \int_{B(1/2) \setminus D(x, r_n)} \exp(-n \mathrm{Leb}(D(x, r_n) \cup D(y, r_n))) dy dx.$$

Splitting the inner integral according to whether or not $y \in D(x, 2r)$ we find $\mathbb{E}[N_n(N_n - 1)]$ is at most

$$\begin{aligned} &(n/4)^2 \exp(-2nr_n^2) + (n^2/4)\pi(4r_n^2) \exp(-nr_n^2) \\ &\qquad = (\mathbb{E}[N_n])^2 + O(n^{1-\alpha} \log n), \end{aligned}$$

so that $\limsup(\mathbb{E}[N_n^2]/(\mathbb{E}[N_n])^2) \leq 1$. Hence, $\mathrm{Var}(N_n/\mathbb{E}[N_n]) \to 0$ so $N_n/\mathbb{E}[N_n] \to 1$ in probability. Thus $\mathbb{P}[N_n = 0] \to 0$, and if $N_n > 0$ then $\delta_n = 0$. The result follows. □.

Corollary 2.3. *Given $\varepsilon \in (0, 1)$ we have* $\mathbb{P}[n\pi\rho_n^2/\log n > 1 - \varepsilon] \to 1$.

Proof. Set $r_n = ((1 - \varepsilon) \log n/(n\pi))^{1/2}$, so $n\pi r_n^2/\log n = 1 - \varepsilon$. Let δ_n be the minimum degree of $G(\mathcal{P}_n, r_n)$. If the minimum degree of a graph of order greater than 1 is zero, then it is not connected; hence

$$\begin{aligned} \mathbb{P}[n\pi\rho_n^2/\log n < 1 - \varepsilon] &= \mathbb{P}[G(\mathcal{P}_n, r_n) \in \mathcal{K}] \\ &\leq \mathbb{P}[\delta_n > 0] + \mathbb{P}[\mathcal{P}_n(B(1)) \leq 1], \end{aligned}$$

which tends to zero by Theorem 2.2. □

To complete the proof of Theorem 2.1, it suffices to prove the following:

Theorem 2.4. *Suppose $(r_n)_{n \in \mathbb{N}}$ is such that*

$$n\pi r_n^2/\log n = \alpha > 1, \qquad \forall n \geq 2. \tag{2.2}$$

Then $\mathbb{P}[G(\mathcal{P}_n, r_n) \in \mathcal{K}] \to 1$.

The proof of this requires a series of lemmas. It proceeds by discretization of space.

Assume $d = 2$ and r_n is given, satisfying (2.2). Let $\varepsilon \in (0, 1/9)$ be chosen in such a way that

$$(1-\varepsilon)\alpha((1-3\varepsilon)^2 - 2\varepsilon) > 1+\varepsilon. \tag{2.3}$$

Divide $B(1)$ into squares of side εr_n; actually we should use squares of side $1/\lfloor(1/\varepsilon r_n)\rfloor$ so they fit exactly, but to ease notation we shall ignore this minor technicality and assume/pretend that $1/(\varepsilon r_n)$ is an integer for all n.

Let $\mathcal{L}_n$ be the set of centres of these squares (a finite lattice). Then $|\mathcal{L}_n| = \Theta(n/\log n)$.

List the squares as $Q_i, 1 \leq i \leq |\mathcal{L}_n|$, and the corresponding centres of squares (i.e., the elements of $\mathcal{L}_n$) as $q_i, 1 \leq i \leq |\mathcal{L}_n|$.

Given $U \subset \mathcal{P}_n$, let us say $q_i \in \mathcal{L}_n$ is U*-occupied* if $U \cap Q_i \neq \varnothing$. Let $\mathcal{O}_n(U)$ be the set of sites $q_i \in \mathcal{L}_n$ that are U-occupied.

Lemma 2.5. *Let $U \subset \mathcal{P}_n$ be such that $G(U, r_n) \in \mathcal{K}$. Then also $G(\mathcal{O}_n(U), r_n(1+2\varepsilon)) \in \mathcal{K}$.*

Proof. Suppose $x, y \in U$ with $\{x, y\}$ an edge of $G(U, r_n)$. Choose $q_i, q_j \in \mathcal{L}_n$ with $x \in Q_i$ and $y \in Q_j$. By the triangle inequality

$$\begin{aligned}\|q_i - q_j\| &\leq \|q_i - x\| + \|x - y\| + \|y - q_j\| \\ &\leq r_n\varepsilon + r_n + r_n\varepsilon = r_n(1+2\varepsilon),\end{aligned}$$

so either $i = j$ or $\{q_i, q_j\}$ is an edge of $G(\mathcal{O}_n(U), r_n(1+2\varepsilon))$.

Given $q_k, q_\ell \in \mathcal{O}_n(U)$, pick $u \in U \cap Q_k$ and $v \in U \cap Q_\ell$. Then there is a path in $G(U, r_n)$ from u to v and by the above, taking the box centres of the successive points in this path provides a path in $G(\mathcal{O}_n(U), r_n(1+2\varepsilon))$ from q_k to q_ℓ. Hence $G(\mathcal{O}_n(U), r_n(1+2\varepsilon))$ is connected. □

Let $\mathcal{A}_{n,m}$ denote the set of $\sigma \subset \mathcal{L}_n$ with m elements such that $G(\sigma, r_n(1+2\varepsilon)) \in \mathcal{K}$ (sometimes called 'lattice animals').

Let $\mathcal{A}^2_{n,m}$ be the set of $\sigma \in \mathcal{A}_{n,m}$ such that $\text{dist}(\sigma, \partial B(1)) > 2r_n$, i.e. all elements of σ are distant at least $2r_n$ from the boundary of $B(1)$.

Let $\mathcal{A}^1_{n,m}$ be the set of $\sigma \in \mathcal{A}_{n,m}$ such that σ is distant less than $2r_n$ from *just one edge* of $B(1)$.

Let $\mathcal{A}^0_{n,m} := \mathcal{A}_{n,m} \setminus (\mathcal{A}^2_{n,m} \cup \mathcal{A}^1_{n,m})$, the set of $\sigma \in \mathcal{A}_{n,m}$ such that σ is distant less than $2r_n$ from *two edges* of $B(1)$ (i.e. near a corner of $B(1)$).

The counting argument in the next lemma is sometimes called a *Peierls argument*.

Lemma 2.6. *Given $m \in \mathbb{N}$, there is constant $C = C(m)$ such that for all n,*

$$|\mathcal{A}_{n,m}| \leq C(n/\log n),$$
$$|\mathcal{A}^1_{n,m}| \leq C(n/\log n)^{1/2}, \qquad |\mathcal{A}^0_{n,m}| \leq C.$$

Proof. Fix m. Consider how many ways there are to choose $\sigma \in \mathcal{A}_{n,m}$.

There are at most r_n^{-2} choices, and hence $O(n/\log n)$ choices, for the first element of σ in the lexicographic ordering. Having chosen the first element of σ, there are a bounded number of ways to choose the rest of σ.

Consider how many ways there are to choose $\sigma \in \mathcal{A}^1_{n,m}$. In this case there are $O(r_n^{-1}) = O((n/\log n)^{1/2})$ ways to choose the first element of σ (distant at most $2r_n$ from the boundary of $[0,1]^2$), and then a bounded number of ways to choose the rest of σ.

Finally consider how many ways there are to choose $\sigma \in \mathcal{A}^0_{n,m}$. In this case there are $O(1)$ ways to choose the first element of σ, and then a bounded number of ways to choose the rest of σ. $\square$

For $n \in \mathbb{N}$, let $\mathcal{K}_n(\mathcal{P}_n)$ be the collection of vertex sets of the components of $G(\mathcal{P}_n, r_n)$ (a partition of $\mathcal{P}_n$). Given $\sigma \subset \mathcal{L}_n$, let E_σ be the event that there exists $U \in \mathcal{K}_n(\mathcal{P}_n)$ such that $\mathcal{O}_n(U) = \sigma$.

Lemma 2.7. *Assume r_n satisfy (2.2) and ε has been chosen to satisfy (2.3). Let $m \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ with $n \geq 2$,*

$$\sup_{\sigma \in \mathcal{A}^2_{n,m}} \mathbb{P}[E_\sigma] \leq n^{-(1+\varepsilon)}. \tag{2.4}$$

Also

$$\sup_{\sigma \in \mathcal{A}^1_{n,m}} \mathbb{P}[E_\sigma] \leq n^{-(1+\varepsilon)/2} \tag{2.5}$$

and

$$\sup_{\sigma \in \mathcal{A}^o_{n,m}} (\mathbb{P}[E_\sigma]) \leq n^{-(1+\varepsilon)/4}. \tag{2.6}$$

Proof. Given $\sigma \in \mathcal{A}^2_{n,m}$, let q_i (respectively q_j) be the lexicographically first (resp. last) element of σ. Let D^-_σ be the part of $D(q_i, r_n(1-3\varepsilon))$ lying to the left of Q_i. Let D^+_σ be the part of $D(q_j, r_n(1-3\varepsilon))$ lying to the right of Q_j.

We claim that if E_σ occurs, then $\mathcal{P}_n(D^-_\sigma) = 0$ and $\mathcal{P}_n(D^+_\sigma) = 0$. Indeed, if E_σ occurs and $\mathcal{P}_n(D^-_\sigma) \neq 0$, we can choose $z \in \mathcal{P}_n \cap D^-_\sigma$, and also $U \in \mathcal{K}_n(\mathcal{P}_n)$ such that $\mathcal{O}_n(U) = \sigma$, and also $y \in U \cap Q_i$. Then

$$\|z - y\| \leq \|z - q_i\| + \|q_i - y\| \leq r_n(1-3\varepsilon) + \varepsilon r_n < r_n,$$

so also $y \in U$, but then taking k such that $z \in q_k$, we have $k \in \mathcal{O}_n(U)$ but also q_k to the left of q_i, a contradiction. This shows the first part of the claim, and we can argue similarly for D^+_σ. By the claim,

$$\begin{aligned}
\mathbb{P}[E_\sigma] &\leq \mathbb{P}[\mathcal{P}_n(D^-_\sigma \cup D^+_\sigma) = 0] \\
&\leq \exp(-n[\pi(r_n(1-3\varepsilon))^2 - 2\varepsilon r_n^2]) \\
&\leq \exp\left[-(\alpha \log n)\left((1-3\varepsilon)^2 - 2\varepsilon\right)\right]
\end{aligned}$$

By (2.3), this is less than $n^{-1-\varepsilon}$, completing the proof of (2.4).

To prove (2.5). Take $\sigma \in \mathcal{A}^1_{n,m}$. Consider just the case where σ is near to the left edge of $B(1)$. Define D^+_σ as above. Then

$$\begin{aligned}
&\mathbb{P}[E_\sigma] \leq \mathbb{P}[\mathcal{P}_n(D^+_\sigma) = 0] \\
&\leq \exp(-(n/2)\pi(r_n(1-3\varepsilon))^2 - 2\varepsilon r_n^2) \\
&\leq \exp\left[-\left(\frac{\alpha \log n}{2}\right)((1-3\varepsilon)^2 - 2\varepsilon)(1-\varepsilon)\right]
\end{aligned}$$

and by (2.3) this is less than $n^{-(1+\varepsilon)/2}$ completing the proof of (2.5).

The proof of (2.6) is similar. $\square$

Lemma 2.8. *Let $m \in \mathbb{N}$. Then $\mathbb{P}[\exists U \in \mathcal{K}_n(\mathcal{P}_n) : |\mathcal{O}_n(U)| = m] \to 0$ as $n \to \infty$.*

Proof. By Lemma 2.5, if $U \in \mathcal{K}_n(\mathcal{P}_n)$ with $|\mathcal{O}_n(U)| = m$, then $\mathcal{O}_n(U) \in \mathcal{A}_{n,m}$. Hence by Lemma 2.7,

$$\begin{aligned}\mathbb{P}[\exists U \in \mathcal{K}_n(\mathcal{P}_n) : |\mathcal{O}_n(U)| = m] \\ \leq \sum_{\sigma \in \mathcal{A}_{n,m}} \mathbb{P}(E_\sigma) \\ \leq |\mathcal{A}^2_{n,m}| n^{-(1+\varepsilon)} + |\mathcal{A}^1_{n,m}| n^{-(1+\varepsilon)/2} \\ + |\mathcal{A}^0_{n,m}| n^{-(1+\varepsilon)/4}\end{aligned}$$

and using Lemma 2.6 we find that this tends to zero. $\square$

For $U \subset \mathbb{R}^2$ define

$$U^r := \cup_{x \in U} D(x, r).$$

Write diam_∞ for diameter in the ℓ_∞ norm. For any bounded connected $A \subset \mathbb{R}^2$, $\mathrm{diam}_\infty(A)$ is the smallest possible side length of a rectilinear square containing A.

Lemma 2.9. *Let $Q = [-1/2, 1/2]^2$, $Q^o = (-1/2, 1/2)^2$ and $\partial Q = Q \setminus Q^o$. Let $r > 0$ and suppose U, V are disjoint finite nonempty subsets of Q^o. Suppose the sets U^r and V^r are connected and $U^r \cap V^r = \varnothing$.*

Then there exists a connected set $\Gamma \subset Q^o \cap \partial(U^r)$ with

$$\mathrm{diam}_\infty(\Gamma) \geq \min(\mathrm{diam}_\infty(U^r), \mathrm{diam}_\infty(V^r)). \tag{2.7}$$

Proof. Given $x \in U$, let us define an *exposed arc* of the circle $\partial D(x, r)$ to be a portion of this circle within Q^o that is not covered by any of the other disks, i.e. a connected component of $Q^o \cap (\partial D(x, r)) \setminus \cup_{y \in U \setminus \{x\}} D(y, r)$.

Then $\partial_Q(U^r)$ (the boundary of U^r relative to Q) consists of all the exposed arcs of the circles $\partial D(x, r), x \in U$, together with some vertices of degree 2 (wherever two exposed arcs meet) or degree 1 (wherever an exposed arc meets ∂Q). The exposed arcs and vertices can be seen as a finite plane graph with all vertices of degree 1 or 2.

Such a graph must split into a finite collection of cycles, each of which is a Jordan curve, along with some curves (paths) which start and end at points in ∂Q. These cycles and curves are all disjoint from each other. Denote these cycles and curves by $\Gamma_1, \ldots, \Gamma_m$.

The set V^r lies in a single component of the complement of $\partial_Q(U^r)$ and the boundary of this component (relative to Q) is one of the curves $\Gamma_1, \ldots, \Gamma_m$, without loss of generality Γ_1. Then taking $\Gamma = \Gamma_1$, we have that any continuous path in Q^o from V^r to U^r must pass through Γ.

If $\mathrm{diam}_\infty(\Gamma) < \min(\mathrm{diam}_\infty(U^r), \mathrm{diam}_\infty(V^r))$, then we can find a closed rectilinear square S containing Γ of side $\mathrm{diam}_\infty(\Gamma)$, but also can find $x \in U^r \cap Q^o \setminus S$ and $y \in V^r \cap Q^o \setminus S$. But then we could find a continuous path in Q^o from x to y avoiding S, contradicting our earlier conclusion that any path from U^r to V^r must pass through Γ. Therefore (2.7) holds. $\square$

Given $K \in \mathbb{N}$, let $F_K(n)$ be the event that there exist distinct $U, V \in \mathcal{K}_n(\mathcal{P}_n)$ such that $\min(|\mathcal{O}_n(U)|, |\mathcal{O}_n(V)|) \geq K$.

Lemma 2.10. *There exists $K \in \mathbb{N}$ such that $\mathbb{P}[F_K(n)] \to 0$ as $n \to \infty$.*

Proof. Suppose $F_K(n)$ occurs. Then there exist distinct $U, V \in \mathcal{K}_n(\mathcal{P}_n)$, such that $\min(|\mathcal{O}_n(U)|, |\mathcal{O}_n(V)|) \geq K$. Let $U' := U^{r_n/2}$, and $V' := V^{r_n/2}$. By Lemma 2.9, there is a connected set $\Gamma \subset \partial U' \cap (-\frac{1}{2}, \frac{1}{2})^2$ with $\mathrm{diam}_\infty(\Gamma) \geq \min(\mathrm{diam}_\infty(U'), \mathrm{diam}_\infty(V'))$.

Let τ be the set of $q_i \in \mathcal{L}_n$ such that $Q_i \cap \Gamma \neq \varnothing$.

Then τ is $*$-connected in $\mathcal{L}_n$, i.e. for any two sites x, y in τ, there is a path $(x_0, x_1, \ldots, x_k)$ with $x_0 = x$, $x_k = y$ and $\|x_i - x_{i-1}\|_\infty = \varepsilon r_n$ for $1 \leq i \leq k$.

Also, for each $q_i \in \tau$ we claim $\mathcal{P}_n(Q_i) = 0$. Indeed, any such Q_i contains part of the boundary of U', so if there were a point of $\mathcal{P}_n$ in Q_i it would be distant at most $r_n((1/2) + 2\varepsilon)$ from U and therefore would actually be in U so Q_i would *not* include any of the boundary of U', a contradiction.

Next we claim the isoperimetric inequality $|\tau| \geq K^{1/2}$. To see this note that $\cup_{i:q_i \in \mathcal{O}_n(U)} Q_i$ is contained in U', and therefore with $\mathrm{Leb}(\cdot)$ denoting area,

$$\mathrm{Leb}(U') \geq \mathrm{Leb}(\cup_{i:q_i \in \mathcal{O}_n(U)} Q_i) \geq K\varepsilon^2 r_n^2.$$

Also $\mathrm{diam}_\infty(U') \geq (\mathrm{Leb}(U'))^{1/2}$, sinc U' is contained in a square of side $\mathrm{diam}_\infty(U')$.

Hence $\mathrm{diam}_\infty(U') \geq K^{1/2}\varepsilon r_n$. Likewise $\mathrm{diam}_\infty(V') \geq K^{1/2}\varepsilon r_n$ and hence $\mathrm{diam}_\infty(\Gamma) \geq K^{1/2}\varepsilon r_n$. Since $|\tau| \geq \mathrm{diam}_\infty(\Gamma)/(\varepsilon r_n)$, we have the claim.

Let $\mathcal{A}'_{n,m}$ be the set of $*$-connected subsets of $\mathcal{L}_n$ with m elements. By a Peierls argument related to the proof of Lemma 2.6 (see also [3, Lemma 9.3]), there are finite constants γ and C (we can take $\gamma = 2^8$ for example) such that

$$|\mathcal{A}'_{n,m}| \leq C(n/\log n)\gamma^m.$$

Set

$$\begin{aligned} \phi_n := \mathbb{P}[\mathcal{P}_n(Q_i) = 0] &= \exp(-n(\varepsilon r_n)^2) \\ &= \exp[-\varepsilon^2(\alpha/\pi)(\log n)], \end{aligned}$$

where the last line comes from (2.2). Then

$$\begin{aligned} \mathbb{P}[F_K(n)] &\leq \sum_{m \geq K^{1/2}} C(n/\log n)\gamma^m \phi_n^m \\ &\leq 2C(n/\log n)(\gamma n^{-\varepsilon^2/\pi})^{K^{1/2}} \\ &= 2C\gamma^{K^{1/2}} n^{1-\varepsilon^2\pi^{-1}K^{1/2}}/\log n \end{aligned}$$

which tends to zero provided K is chosen large enough so that $(\varepsilon^2/\pi)K^{1/2} > 1$. □

Proof of Theorem 2.4. Choose $K \in \mathbb{N}$ as in Lemma 2.10. Then by Lemma 2.5 we have that

$$\begin{aligned} \mathbb{P}[G(\mathcal{P}_n, r_n) \notin \mathcal{K}] &\leq \mathbb{P}[\exists U, V \in \mathcal{K}_n(\mathcal{P}_n), U \neq V] \\ &\leq \left(\sum_{m=1}^{K} \mathbb{P}[\exists U \in \mathcal{K}_n(\mathcal{P}_n), |\mathcal{O}_n(U)| = m]\right) \\ &\qquad + \mathbb{P}[F_K(n)]. \end{aligned}$$

By Lemmas 2.8. and 2.10, this tends to zero. □

3 Percolative ingredients

To prepare for proving results about large components of random geometric graphs, we recall (without proof) some facts about percolation theory, which is the study of connectivity properties of random sets in space.

3.1 Site percolation on a lattice

The triangular lattice is the the graph $G(\mathbb{T}, 1)$, where $\mathbb{T}$ is the set in $\mathbb{R}^2$ defined by $\mathbb{T} := \{m(1,0) + n(1/2, \sqrt{3}/2) : m, n \in \mathbb{Z}\}$.

Given $p \in [0,1]$, let $Z^p = (Z^p_x, x \in \mathbb{T})$ be a family of mutually independent Bernoulli(p) random variables. Let

$$\mathcal{O}(Z^p) := \{x \in \mathbb{T} : Z^p_x = 1\}.$$

Sites in $\mathcal{O}(Z^p)$ are called **open** or **occupied**.

Site percolation theory is concerned with the components of the graph

$$G(\mathcal{O}(Z^p), 1),$$

in partaicular the infinite or large components.

3.2 k-dependent percolation

Let $k \in \mathbb{N}$. We say $Z = (Z_x, x \in \mathbb{T})$ is a (weakly) **k-dependent Bernoulli random field** on $\mathbb{T}$, if it is a collection of $\{0,1\}$-valued random variables in the same probability space, such that for any finite $A \subset \mathbb{T}$ with $\|x - y\| > k$ for any distinct $x, y \in A$, the random variables $(Z_x, x \in A)$ are mutually independent. Let $\mathcal{O}(Z) = \{x \in \mathbb{T} : Z(x) = 1\}$ (the set of 'open' or 'occupied' sites)..

Given $m, n \in \mathbb{N}$, let $LR(Z, m, n)$ be the event that there is a path in $G(\mathcal{O}(Z), 1)$ from the left edge to the right edge of the rectangle

$$R(m,n) := [0, n-1] \times [0, (m-1)\sqrt{3}/2].$$

Lemma 3.1. *Let $k \in \mathbb{N}$. There exists $p^*_c(k) \in (0,1)$ such that for* **any** *k-dependent Bernoulli random field Z on $\mathbb{T}$ with $\mathbb{P}[Z_x = 1] \geq p^*_c(k)$ for all $x \in \mathbb{T}$,*

$$\mathbb{P}[\{LR(Z, m, n)\}] \leq n2^{-m} \qquad \forall n, m \in \mathbb{N}.$$

Proof. (Sketch) If $LR(Z, m, n)$ fails, there is a vertical crossing of $R(n, m)$ in $G(\mathcal{O}(Z)^c, 1)$.

Hence for some path γ in $G(\mathbb{T}, 1)$ of length $\ell \geq m$ starting on $[0, n-1] \times \{0\}$, we have $\gamma \subset \mathcal{O}(Z)^c$.

The number of such paths γ is at most $n\beta^\ell$ for some constant $\beta < \infty$.

By taking $\mathbb{P}[Z_x = 1]$ close enough to 1, we can arrange for $\mathbb{P}[\gamma \subset \mathcal{O}(Z)^c] \leq (1/(2\beta))^\ell$.

Then the result follows from the union bound. □

3.3 Continuum percolation

We consider the components of the infinite graph $G(\mathcal{H}_\lambda; 1)$. Equivalently, consider the connected components of the set $\mathcal{H}^+_\lambda$, where for $\mathcal{X} \subset \mathbb{R}^2$ we set

$$\mathcal{X}^+ := \mathcal{X}^{1/2} = \cup_{x \in \mathcal{X}} D(x, \frac{1}{2}).$$

Let $\mathcal{H}_{\lambda,0} := \mathcal{H}_\lambda \cup \{\mathbf{o}\}$, and

$$\mathcal{C}_0(\lambda) := \{x \in \mathcal{H}_{\lambda,0} : \mathbf{o} \leftrightarrow x \text{ in } G(\mathcal{H}_{\lambda,0}, 1)\}.$$

For $k \in \mathbb{N}$, set $p_k(\lambda) := \mathbb{P}[|\mathcal{C}_0(\lambda)| = k]$.

Define the **continuum percolation probability** $p_\infty(\lambda)$ by

$$p_\infty(\lambda) := \mathbb{P}[|\mathcal{C}_0(\lambda)| = \infty] = 1 - \sum_{k=1}^{\infty} p_k(\lambda).$$

Exercise 3.2. Suppose $0 < \lambda < \lambda'$. Show that $p_\infty(\lambda) \leq p_\infty(\lambda')$.

Define the *critical value* λ_c by

$$\lambda_c = \inf\{\lambda > 0 : p_\infty(\lambda) > 0\}.$$

Theorem 3.3. *We have* $0 < \lambda_c < \infty$.

Simulation studies indicate that $\lambda_c \approx 1.44$.

It is known that $p_\infty(\lambda_c) = 0$. It is not known if this holds in all dimensions.

3.4 Uniqueness of the infinite cluster

Fix $\lambda > 0$ and let N be the number of infinite components of the graph $G(\mathcal{H}_\lambda; 1)$.

Exercise 3.4. Show that if $p_\infty(\lambda) = 0$, then $\mathbb{P}[N = 0] = 1$.

Theorem 3.5. *Suppose* $p_\infty(\lambda) > 0$. *Then* $\mathbb{P}[N = 1] = 1$.

In other words, two things happen almost surely when $p_\infty(\lambda) > 0$:

(i) There is an infinite component of $G(\mathcal{H}_\lambda, 1)$.

(ii) There is only one such component.

3.5 The Harris-FKG inequality

We say a real-valued function f, defined on locally finite point configurations $\mathcal{X} \subset \mathbb{R}^d$, is **increasing** if $f(\mathcal{X}) \leq f(\mathcal{Y})$ whenever $\mathcal{X} \subset \mathcal{Y}$. We say f is **decreasing** if $-f$ is increasing. Given $\lambda > 0$, we say E is an increasing (resp. decreasing) event on $\mathcal{H}_\lambda$ if $\mathbf{1}_E$ is an increasing (resp. decreasing) function of $\mathcal{H}_\lambda$.

Theorem 3.6 (Harris-FKG inequality)**.** *Suppose f, g are measurable bounded increasing real-valued functions defined on locally finite point configurations in $\mathbb{R}^2$. Then* $\mathrm{Cov}(f(\mathcal{H}_\lambda), g(\mathcal{H}_\lambda)) \geq 0$, *i.e.*

$$\mathbb{E}[f(\mathcal{H}_\lambda)g(\mathcal{H}_\lambda)] \geq \mathbb{E}[f(\mathcal{H}_\lambda)]\mathbb{E}[g(\mathcal{H}_\lambda)].$$

The same inequality holds if f and g are both decreasing.

Corollary 3.7 (Square Root trick)**.** *Let $\lambda > 0$, $k \in \mathbb{N}$, $\varepsilon \in (0,1)$. Suppose for $i = 1, \ldots, k$ we have increasing events A_i defined on $\mathcal{H}_\lambda$. such that $\mathbb{P}[\cup_{i=1}^k A_i] > 1 - \varepsilon$.*

Then $\max_{1 \leq i \leq k} \mathbb{P}[A_i] > 1 - \varepsilon^{1/k}$.

Proof. Set $M = \max_{1 \le i \le k} \mathbb{P}[A_i]$. The events A_i^c are all decreasing, so by the Harris-FKG inequality

$$\varepsilon > \mathbb{P}[\cap_{i=1}^k A_i^c] \ge \prod_{i=1}^k \mathbb{P}[A_i^c] \ge (1-M)^k,$$

so that $1 - M < \varepsilon^{1/k}$ and $M > 1 - \varepsilon^{1/k}$. □

Remark. Often in applications of the Square Root trick, the sets A_i all have the same probability.

4 The largest component

In this section we aim to prove a Poissonized version of Theorem 1.4, concerning the theromodynamic limit $nr_n^2 \to \lambda$. In fact we just take $nr_n^2 = \lambda$.

For any finite graph G, let $L_j(G)$ denote the order of its jth-largest component, that is, the jth-largest of the orders of its components, or zero if it has fewer than j components.

Recall that $\mathcal{H}_{\lambda,s}$ denotes a homogeneous Poisson process of intensity λ in $B(s) = [-s/2, s/2]^2$, and $\mathcal{P}_n := \mathcal{H}_{n,1}$. By the representation (1.1), $\mathcal{P}_n = \{\xi_1, \dots, \xi_{N_n}\}$ so

$$\begin{aligned} L_1(G(\mathcal{P}_n, r_n)) &= L_1(G(\{r_n^{-1}\xi_1, \dots, r_n^{-1}\xi_{N_n}\}, 1)) \\ &= L_1(G(\mathcal{H}_{nr_n^2}, 1)) = L_1(G_{\lambda,s}, 1), \end{aligned}$$

where we set $\lambda = nr_n^2$ and $s = r_n^{-1}$.

Therefore we shall consider $L_1(G(\mathcal{H}_{\lambda,s}, 1))$ (and also $L_2(G(\mathcal{H}_{\lambda,s}, 1))$) as $s \to \infty$ with λ fixed. From now on, instead of (1.1) we take $\mathcal{H}_{\lambda,s} := \mathcal{H}_\lambda \cap B(s)$.

4.1 The subcritical case

Theorem 4.1. *Suppose $\lambda > 0$ with $p_\infty(\lambda) = 0$. Then*

$$s^{-2} L_1(G(\mathcal{H}_{\lambda,s}; 1)) \xrightarrow{P} 0 \qquad \text{as} \quad s \to \infty. \tag{4.1}$$

Proof. Suppose $p_\infty(\lambda) = 0$. For any locally finite $\mathcal{X} \subset \mathbb{R}^2$ and $x \in \mathcal{X}$, let $C_x(\mathcal{X})$ denote the vertex set of the component of $G(\mathcal{X}, 1)$ containing x.

Let $\varepsilon > 0$. Let N_s be the number of $x \in \mathcal{H}_{\lambda,s}$ such that $|C_x(\mathcal{H}_{\lambda,s})| \ge \varepsilon s^2$. If $L_1(G(\mathcal{H}_{\lambda,s}, 1)) \ge \varepsilon s^2$, then $N_s \ge \varepsilon s^2$. Hence by Markov's inequality and the Mecke formula,

$$\begin{aligned} &\mathbb{P}[L_1(G(\mathcal{H}_{\lambda,s}, 1)) \ge \varepsilon s^2] \le (\varepsilon s^2)^{-1} \mathbb{E}[N_x] \\ &= (\varepsilon s^2)^{-1} \int_{B(s)} \mathbb{P}[|C_x(\mathcal{H}_{\lambda,s} \cup \{x\})| \ge \varepsilon s^2] \lambda dx \\ &\le (\lambda/(\varepsilon s^2)) \int_{B(s)} \mathbb{P}[|C_x(\mathcal{H}_\lambda \cup \{x\})| \ge \varepsilon s^2] dx \\ &= (\lambda/\varepsilon) \sum_{k \ge \varepsilon s^2} p_k(\lambda) \end{aligned}$$

which tends to zero as $s \to \infty$. Therefore $s^{-2} L_1(G(\mathcal{H}_{\lambda,s}, 1)) \xrightarrow{P} 0$. □

Exercise 4.2. Suppose $\lambda > \lambda_c$, and let $k, K \in \mathbb{N}$. Let $N_k(s)$ be the number of $x \in \mathcal{H}_{\lambda,s}$ such that $|C_x(\mathcal{H}_{\lambda,s})| = k$, and set $N_{\leq K} := \sum_{k=1}^{K} N_k(s)$.

(i) Use the Mecke formula to show that

$$\lim_{s\to\infty} s^{-2}\mathbb{E}[N_k(s)] = \lambda p_k(\lambda),$$

and that $\lim_{s\to\infty} \mathrm{Var}[s^{-2}N_k(s)] = 0$.

(ii) Deduce that $(\lambda s^2)^{-1}N_{\leq K}(s) \xrightarrow{P} \sum_{k=1}^{K} p_k(\lambda)$.

(iii) Using (1.1), show that $L_1(G(\mathcal{H}_{\lambda,s}, 1)) \leq \max(K, N_{\lambda s^2} - N_{\leq K}(s))$.

(iv) *Let $\varepsilon > 0$. Deduce that as $s \to \infty$,*

$$\mathbb{P}[(\lambda s^2)^{-1}L_1(G(\mathcal{H}_{\lambda,s}, 1)) > p_\infty(\lambda) + \varepsilon] \to 0$$

(v) *Show that as $s \to \infty$,*

$$\begin{aligned}&\mathbb{P}[(\lambda s^2)^{-1}L_1(G(\mathcal{H}_{\lambda,s}, 1))\\ &\quad + (\lambda s^2)^{-1}L_2(G(\mathcal{H}_{\lambda,s}, 1)) > p_\infty(\lambda) + \varepsilon] \to 0\end{aligned}$$

4.2 Renormalization

We now write $D_r(x)$ for the disk $D(x,r)$ and D_r for $D(\mathbf{o},r)$. Also let $S_r = [-r,r]^2$, and let $\mathbf{e} := (1,0)$. Given $\lambda, K, L, M \in (0,\infty)$ with $L > K, M > 2K$, define the following events:

- $U_{K,L,\lambda}$ is the event that there is a unique component of $\mathcal{H}_\lambda^+ \cap D_L$ that meets both D_K and ∂D_L.
- $F_{K,M,\lambda}$ is the event that there is a path in $\mathcal{H}_\lambda^+ \cap D_{3M}$ from D_K to $D_K(M\mathbf{e})$.

Proposition 4.3. *Suppose $p_\infty(\lambda) > 0$ and let $\varepsilon \in (0,1)$. There exist constants $K > 0$ and $M > 3K$ such that $\mathbb{P}[U_{K,M/3,\lambda}] > 1-\varepsilon$ and $\mathbb{P}[F_{K,M,\lambda}] > 1-\varepsilon$.*

Proof (sketch). We adapt an argument of
Duminil-Copin, Sidoravicius and Tassion [1]. Let $\varepsilon_1 = (1/3)\varepsilon^{32} < \varepsilon$. Choose K such that $\mathbb{P}[D_K \leftrightarrow \infty] > 1-\varepsilon_1$. Using uniqueness (Theorem 3.5) we can and do choose $n_1 \in \mathbb{N}$ with $n_1 > K$ such that

$$\mathbb{P}[U_{K,n,\lambda}] \geq 1-\varepsilon, \quad \forall n \geq n_1. \tag{4.2}$$

For $n \geq n_1$, and $0 \leq \alpha \leq \beta \leq n$, let

$$E_n(\alpha,\beta) = \{D_K \leftrightarrow \{n\} \times [\alpha,\beta] \text{ in } \mathcal{H}_\lambda^+ \cap S_n\}.$$

As $\mathbb{P}[D_K \leftrightarrow \partial S_n] \geq \mathbb{P}[D_K \leftrightarrow \infty] > 1-\varepsilon_1$, using the square root trick we can show

$$\mathbb{P}[E_n(0,n)] > 1-\varepsilon_1^{1/8}.$$

Given n, choose $\alpha_n \in (0,n)$ such that

$$\mathbb{P}[E_n(0,\alpha_n)] = \mathbb{P}[E_n(\alpha_n,n)].$$

Can do this since $g(\alpha) := \mathbb{P}[E_n(0,\alpha)] - \mathbb{P}[E_n(\alpha,n)]$ is continuous in α with $g(0) < 0, g(n) > 0$.

Since $E_n(0,n) = E_n(0,\alpha_n) \cup E_n(\alpha_n,n)$, using the square root trick again yields

$$\mathbb{P}[E_n(\alpha_n,n)] = \mathbb{P}[E_n(0,\alpha_n)] > 1-\varepsilon_1^{1/16}. \tag{4.3}$$

Using the square root trick yet again yields

$$\max(\mathbb{P}[E_n(0,\alpha_n/2)], \mathbb{P}[E_n(\alpha_n/2,\alpha_n)]) > 1-\varepsilon_1^{1/32}.$$

Choose either $y_n = \alpha_n/4$ or $y_n = 3\alpha_n/4$, so

$$\mathbb{P}[E_n(y_n-\alpha_n/4, y_n+\alpha_n/4)] > 1-\varepsilon_1^{1/32}. \tag{4.4}$$

Set $n_2 = 3n_1$. We claim that there exists $N \geq n_2$ such that $\alpha_{3N} \leq 4\alpha_N$. Indeed, if this were not true then we would have for all $k \geq 1$ that $\alpha_{3^k n_2} \geq 4^k \alpha_{n_2}$, but since $\alpha_n \leq n$ for all n, this would imply $3^k n_2 \geq 4^k \alpha_{n_2}$ so that $(4/3)^k \leq n_2/\alpha_{n_2}$ for all k, which is not true (here we use the fact that $\alpha_{n_2} > 0$), justifying the claim.

Choose (deterministic) $N \geq n_2$ such that $\alpha_{3N} \leq 4\alpha_N$. Then by (4.2)

$$\min(\mathbb{P}[U_{K,N,\lambda}], \mathbb{P}[U_{K,3N,\lambda}]) > 1-\varepsilon_1, \tag{4.5}$$

and by (4.3) and (4.4), setting $\varepsilon_2 := \varepsilon_1^{1/32}$,

$$\begin{aligned}&\min(\mathbb{P}[E_N(\alpha_N, N)],\\ &\mathbb{P}[E_{3N}(y_{3N} - \frac{\alpha_{3N}}{4}, y_{3N} + \frac{\alpha_{3N}}{4})]) > 1 - \varepsilon_2.\end{aligned} \tag{4.6}$$

Now set $x = (2N, y_{3N})$. Let $S_N(x) = S_N + x$. Define the vertical intervals

$$I = \{3N\} \times [y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4],$$

$$J^+ = \{3N\} \times [y_{3N} + \alpha_N, y_{3N} + N],$$
$$J^- = \{3N\} \times [y_{3N} - N, y_{3N} - \alpha_N].$$

Let A^+ be the event that there is a path from $D_K(x)$ to J^+ in $\mathcal{H}_\lambda^+ \cap S'_N$, and let A^- be the event that there is a path from $D_K(x)$ to J^- in $\mathcal{H}_\lambda^+ \cap S_N(x)$. Then $\mathbb{P}[A^+] = \mathbb{P}[A^-] = \mathbb{P}[E_N(\alpha_N, N)]$.

By (4.6) and the union bound,

$$\begin{aligned}&\mathbb{P}[A^+ \cap A^- \cap E_{3N}(y_{3N} - \frac{\alpha_{3N}}{4}, y_{3N} + \frac{\alpha_{3N}}{4})]\\ &> 1 - 4\varepsilon_2 = 1 - \varepsilon.\end{aligned}$$

If the above event holds, then since $\alpha_{3N}/4 \leq \alpha_N$, there is a path in $\mathcal{H}_\lambda^+ \cap S_{3n}$ from D_K to $D_K(x)$. Since $S_{3N} \subset D_{5N} \subset D_{3\|x\|}$, there is hence a path in $\mathcal{H}_\lambda^+ \cap D_{3\|x\|}$ from D_K to $D_K(x)$.

Set $M := \|x\|$. By rotation invariance we therefore have

$$\mathbb{P}[F_{K,M,\lambda}] > 1 - \varepsilon.$$

Also $M \geq 2N \geq 2n_2 = 6n_1$, so $M/3 \geq n_1$ so that $\mathbb{P}[U_{K,M/3,\lambda}] > 1 - \varepsilon$ by (4.2), so the proof is complete. □

4.3 Rectangle crossings

Suppose $R = [a, b] \times [c, d]$ with $a < b$ and $c < d$. We seay a set $S \subset \mathbb{R}^2$ is *1-crossing* (respectively *2-crossing*) for R if there is a continuous path in $S \cap R$ from the left edge of R to the right edge (resp. from the top edge to the bottom edge).

In this section we establish upper bounds for the probability of non-existence of a component of $(\mathcal{H}_{\lambda;s}^+ \cap R)^+$ that is 1-crossing for certain rectangles R. Given $a, b, \lambda > 0$, define the rectangle $R(a, b) := [-a/2, a/2] \times [-b/2, b/2]$, and the event

$$\mathbf{Cr}(\lambda, a, b) = \{\mathcal{H}_\lambda^+ \text{ is } 1-\text{crossing for } R(a, b)\}.$$

Lemma 4.4. *Let $\lambda > \lambda_c$. Then there exists $c > 0$ such that for all large enough t,*

$$1 - \mathbb{P}[\mathbf{Cr}(\lambda, t^2, t)] \leq \exp(-ct)$$

Proof. Define events $U_{K,m} := U_{K,M,\lambda}$ and $F_{K,M} := F_{K,M,\lambda}$ as in subsection 4.2:

- $U_{K,M}$ is the event that there is a unique component of $\mathcal{H}_\lambda^+ \cap D_{M+1}$ that meets both D_K and ∂D_M.
- $F_{K,M}$ is the event that there is a path in $\mathcal{H}_\lambda^+ \cap D_{5M}$ from D_K to $D_K(M\mathbf{e})$.

Let $p_c^*(7)$ be as in Lemma 3.1. Let $\varepsilon = (1 - p_c^*(7))/9$. Using Proposition 4.3, choose $K > 0, M > 3K$ such that $\mathbb{P}[U_{K,M/3}] > 1 - \varepsilon$ and $\mathbb{P}[F_{K,M}] > 1 - \varepsilon$.

For each $x, y \in \mathbb{T}$ with $\|x - y\| = 1$, let U_x denote the event that there is a unique component of $\mathcal{H}_\lambda^+ \cap D_{M/3}(Mx)$ that meets both $D_K(Mx)$ and $\partial D_{M/3}(Mx)$. Let F_{xy} denote the event that there is a path in $\mathcal{H}_\lambda^+ \cap D_{3M}(Mx)$ that meets both $D_K(Mx)$ and $D_K(My)$.

By translation and rotation invariance of $\mathcal{H}_\lambda$, $\mathbb{P}[U_x] > 1 - \varepsilon$ for each x, and $\mathbb{P}[F_{xy}] > 1 - \varepsilon$ for each (x, y).

For $x \in \mathbb{T}$, let us say $X_x = 1$ if event U_x occurs, and also F_{xy} occurs for each of the six $y \in \mathbb{T}$ with $\|y - x\| = 1$; otherwise set $X_x = 0$.

Then $(X_x, x \in \mathbb{T})$ is a 7-dependent Bernoulli random field. For each $x \in \mathbb{T}$, by the union bound

$$\mathbb{P}[X_x = 0] \leq 7\varepsilon < 1 - p_c^*(7).$$

If there is a path in $G(\mathcal{O}(\mathbb{T}), 1)$ that is 1-crossing for the lattice rectangle $R((t^2/M) + 4, (t/M) - 2) \cap \mathbb{T}$ then $\mathbf{Cr}(\lambda, t^2, t)$ occurs. But by Lemma 3.1, the probability of this not occurring is at most

$$(2t^2/M)(1/2)^{t/(2M)}$$

and the result follows. □

Now for $\lambda, a, b > 0$ define the event

$$\mathbf{Cr}^*(\lambda, a, b) = \{(\mathcal{H}_\lambda \cap R(a, b))^+ \text{ is } 1 - \text{crossing for } R(a, b)\},$$

which differs from $\mathbf{Cr}(\lambda, a, b)$ because only disks with centres inside $R(a, b)$ are allowed to be used.

Lemma 4.5. *Suppose $\lambda > \lambda_c$. Then there is a constant $c > 0$ such that for all large enough a,*

$$1 - \mathbb{P}[\mathbf{Cr}^*(\lambda, a, a/3)] \leq \exp(-ca^{1/2}).$$

Proof. Take $\mu \in (\lambda_c, \lambda)$. By the superposition theorem (Theorem 1.6), we may assume that $\mathcal{H}_\lambda$ is obtained as the union of two independent homogeneous Poisson processes $\mathcal{H}_\mu$ and $\mathcal{H}_{\lambda-\mu}$.

Given a, divide $R(a, a/4)$ lengthwise into strips (rectangles) of dimensions $a \times a^{1/2}$. Take alternate strips in the subdivision (to avoid depencences) and denote these by $T_{a,1}, \ldots, T_{a,\nu_a}$ Then ν_a, the number of strips conidered, is $\Theta(a^{1/2})$.

For $1 \leq i \leq \nu_a$, if $\mathcal{H}_\mu^+$ is 1-crossing for $T_{a,i}$, then $(\mathcal{H}_\mu \cap R(a, a/3))^+$ is 1-crossing for the slightly shorter rectangle (denoted $T'_{a,i}$) obtained by moving the left edge of $T_{a,i}$ by 1/2 to the right, and moving the right edge of $T_{a,i}$ by 1/2 to the left

Let $G_{i,a}$ be the event that $\mathcal{H}_\mu^+$ is 1-crossing for $T'_{i,a}$. By Lemma 4.4, for a large we have

$$\mathbb{P}[G_{1,a}^c] \leq \exp(-ca^{1/2}). \tag{4.7}$$

Let $H_{i,a}$ be the event that in addition to event $G_{i,a}$ occurring, there is a continuum path in $(\mathcal{H}_\lambda \cap T_{i,a})^+$ from the left edge to the right edge of $T_{i,a}$. We assert that there is a constant $\delta > 0$, independent of a, such that for all $i \leq \nu_a$ we have

$$P[H_{i,a} | G_{i,a}] \geq \delta. \tag{4.8}$$

Indeed, given a point x on the left edge of $T'_{i,a}$, the probability that there exist two points X, Y of $\mathcal{H}_{\lambda-\mu}$ such that there is a path to the left edge of $T_{i,a}$ through $\{X\mathcal{Y}\}^+$ is bounded away from zero. Likewise for the right edge.

If event $G_{i,a} \cap H_{i,a}$ occurs for any $i \leq \nu_a$, then $\mathbf{Cr}^*(B(a, a/3))$ occurs. Hence by (4.7) and (4.8), we have for all large enough a that

$$\begin{aligned}\mathbb{P}[\mathbf{Cr}^*(R(a, a/3))^c] &\leq \mathbb{P}[\cup_{i=1}^{\nu_a} G_{i,a}^c] \\ &\quad + \mathbb{P}[\cap_{i=1}^{\nu_a} H_{i,a}^c | \cap_{i=1}^{\nu_a} G_{i,a}] \\ &\leq \nu_a \exp(-ca^{1/2}) + (1-\delta)^{\nu_a},\end{aligned}$$

and since $\nu_a = \Theta(a^{1/2})$, this gives us the result. □

4.4 The giant component

We are nearly ready to prove the Poisson version of Theorem 1.4 concerning the giant component, in the supercritical phase.

Proposition 4.6. *Suppose $\lambda > \lambda_c$. Let E_s denote the event that there is a unique component of $\mathcal{H}_{\lambda,s}^+$ having diameter greater than $6s^{1/2}$. Then $\mathbb{P}[E_s] \to 1$ as $s \to \infty$.*

Proof. Divide $B(s)$ into squares of side $s^{1/2}$ (in general these do not exactly fit, so we should really take squares of side $s/\lfloor s^{1/2} \rfloor$ to make them fit, but we ignore this minor issue). Let $R_{s,1}, R_{s,2}, \ldots R_{s,m_s}$ denote the collection of rectangles of aspect ratio 3, obtained by taking a horizontal or vertical line of three of the rectangles in the subdivision. Then $m_s = \Theta(s)$.

Then, by Lemma 4.5, there exists a constant $c > 0$ such that, for large enough s, and for $1 \leq i \leq m_s$,

$$\begin{aligned}&P[G(\mathcal{H}_\lambda \cap R_{s,i}; 1) \text{ is crossing} \\ &\text{the long way for } R_{s,i}] > 1 - \mathrm{e}^{-cs^{1/4}}.\end{aligned} \tag{4.9}$$

Therefore, if I_s denotes the intersection over all $i \leq m_s$ of the events described in (4.9), $P[I_s]$ exceeds $1 - m_s \exp(-cs^{1/4})$. But, on the event I_s, the long-way crossing components of $\mathcal{H}_\lambda^+ \cap R_{a,i}$ must all be part of the same big component of $\mathcal{H}_{\lambda,s}^+$ (since the long-way crossings for rectangles that intersect at right angles must overlap. Also on I_s, no other component can have diameter greater than $6s^{1/2}$ without intersecting this big component. □

We are now finally ready to complete the proof of the Poissonized version of Theorem 1.4, with extra information about the second largest component.

Theorem 4.7. *Suppose $\lambda > \lambda_c$. As $s \to \infty$,*

$$s^{-d} L_1(G(\mathcal{H}_{\lambda,s}; 1)) \xrightarrow{\mathbb{P}} \lambda p_\infty(\lambda). \tag{4.10}$$

Also,

$$s^{-d} L_2(G(\mathcal{H}_{\lambda,s}; 1)) \xrightarrow{\mathbb{P}} 0. \tag{4.11}$$

Proof. Let N_s^* be the number of points of $\mathcal{H}_{\lambda,s}$ lying in components $\mathcal{H}^+_{\lambda,s}$ of diameter at least $6s^{1/2}$. By the Mecke formula (**Exercise!**), as $s \to \infty$ we have

$$s^{-2}\mathbb{E}[N_s^*] \to \lambda p_\infty(\lambda),$$

and moreover

$$s^{-4}\mathbb{E}[N_s^*(N_s^* - 1)] \to \lambda^2 p_\infty(\lambda)^2.$$

Therefore $\mathrm{Var}[s^{-2}N_s^*] \to 0$, so by Chebyshev's inequality $s^{-2}N_s^*$ converges in probability to $\lambda p_\infty(\lambda)$, as $s \to \infty$.

Hence, given $\varepsilon > 0$, we have as $s \to \infty$ that

$$\mathbb{P}[N_s^* \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)] \to 1 \tag{4.12}$$

If the event E_λ, defined in Proposition 4.6, occurs then all those points of $\mathcal{H}_{\lambda,s}$, that lie in components of $\mathcal{H}^+_{\lambda,s}$ of diameter greater than $6s^{1/2}$, must lie in the same component of $G(\mathcal{H}_{\lambda,s}, 1)$. Therefore if also $N_s^* \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)]$ we have $L_1(G(\mathcal{H}_{\lambda,s}, 1)) \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)$. Therefore by (4.12) and Proposition 4.6 we obtain that

$$\mathbb{P}[L_1(G(\mathcal{H}_{\lambda,s}, 1)) \geq s^2\lambda(p_\infty(\lambda) - \varepsilon)] \to 1.$$

Combined with Exercise 4.2(iv), this yields (4.10).

Using Exercise 4.2(v), we obtain that

$$\begin{aligned}\mathbb{P}[s^{-2}(&L_1(G(\mathcal{H}_{\lambda,s})) \\ &+ L_2(G(\mathcal{H}_{\lambda,s}))) > p_\infty(\lambda) + \varepsilon] \to 0,\end{aligned}$$

and therefore $s^{-2}(L_1(G(\mathcal{H}_{\lambda,s})) + L_2(G(\mathcal{H}_{\lambda,s})))$ converges in probability to the same limit as $s^{-2}(L_1(G(\mathcal{H}_{\lambda,s}))$. This implies $s^{-2}L_2(G(\mathcal{H}_{\lambda,s}))$ must converge in probability to zero. □

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