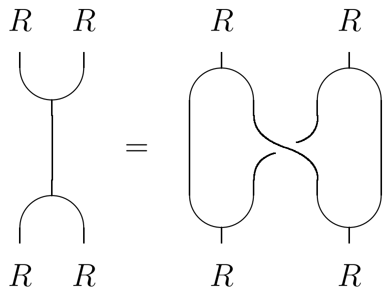
Stefan Ufer

Braided Hopf algebras of triangular type



Braided Hopf algebras of triangular type

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Stefan Ufer

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Erster Gutachter: Prof. Dr. H.-J. Schneider, LMU München
Zweiter Gutachter: Priv. Doz. Dr. P. Schauenburg, LMU München
Dritter Gutachter: Prof. Ch. Kassel, Université Louis Pasteur Strasbourg

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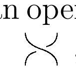
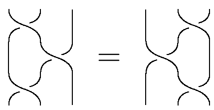
Introduction

The main topic of this thesis are braided Hopf algebras. These objects occur in the structure theory of usual Hopf algebras. Hopf algebras are algebras which are also coalgebras and allow us to turn the tensor product of two representations and the dual of a representation into representations of the Hopf algebra again. The name was chosen in honor of Heinz Hopf who used these algebras when solving a problem on group manifolds in 1941 [12]. During the following years the theory of Hopf algebras was applied for example to affine algebraic groups, to Galois extensions and to formal groups. The interest increased strongly when in the eighties the so-called quantum groups and deformed enveloping algebras were found by Drinfeld [7, 8] and Jimbo [16]. They provided new and non-trivial examples of non-commutative and non-cocommutative Hopf algebras with connections to knot theory, quantum field theory and non-commutative geometry. New results were also obtained in the structure theory of finite-dimensional Hopf algebras and in the classification of certain classes of Hopf algebras and of Hopf algebras with a given dimension.

A braiding on a vector space V is a generalization of the usual flip map $\tau : V \otimes V \to V \otimes V, v \otimes w \mapsto w \otimes v$. It is an automorphism of $V \otimes V$ that satisfies the braid equation

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c).$$

If we interpret the braiding as an operation “interchanging” two tensor factors and represent it by a crossing  , this equation can be visualized by the following picture.



This new way of “interchanging” tensorands allows to generalize the axioms of a usual Hopf algebra, replacing at a certain place the flip map by a braiding. What we get is called a braided Hopf algebra.

Braided Hopf algebras appeared in the structure theory of Hopf algebras when Radford [34] generalized the notion of semi-direct products of groups and of Lie algebras to Hopf algebras. The term "braided Hopf algebra" was introduced by Majid around 1990. Various results for finite-dimensional Hopf algebras were transferred to braided Hopf algebras, for example the famous Nichols-Zoeller theorem and parts of the structure theory (see [44] for a survey). Nevertheless, as one might expect, the theory of braided Hopf algebras is much more complicated than the theory of ordinary Hopf algebras. For example the cocommutative connected case in characteristic zero is well understood in the case of ordinary Hopf algebras (there are only the universal enveloping algebras of Lie algebras), but the knowledge of connected braided Hopf algebras, even for very simple braidings, is quite limited. The connected case is particularly important in the structure theory of pointed Hopf algebras.

The purpose of this thesis is to present new results on braided Hopf algebras of triangular type. These are braided Hopf algebras generated by a finite-dimensional braided subspace of the space of primitive elements (in particular they are connected), such that the braiding fulfills a certain triangularity property. Braidings induced by the quasi-$\mathcal{R}$-matrix of a deformed enveloping algebra are triangular. They yield interesting examples of braided Hopf algebras of triangular type. Another class of triangular braidings are those coming from Yetter-Drinfeld modules over abelian groups. The notion of triangular braidings in this generality is new and was not considered before in the literature.

One of the main results of this thesis is the PBW Theorem 2.2.4 for braided Hopf algebras of triangular type. The concept of PBW bases has its roots in Lie theory, in the famous theorem by Poincaré, Birkhoff, and Witt, which was stated in a first version by Poincaré [32] and improved later by Birkhoff and Witt. If we have a Lie algebra $\mathfrak{g}$, a basis S of $\mathfrak{g}$ and a total order $<$ on S, then this theorem states that the set of all elements of the form

$$s_1^{e_1} \dots s_n^{e_n}$$

with $n \in \mathbb{N}$, $s_1, \dots, s_n \in S, s_1 < s_2 < \dots < s_n$ and $e_i \in \mathbb{N}$ for all $1 \leq i \leq n$ forms a basis of the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra. This basis is an important tool for calculations in the enveloping algebra. A good example is the characterization of primitive elements of $U(\mathfrak{g})$.

In 1958 Shirshov [42] found a basis of the free Lie algebra generated by a set X which consists of standard bracketings of certain words with letters from X. He called these words standard words; we follow Reutenauer and Lothaire when we use the name Lyndon words. Later Lalonde and Ram

[24] showed that if a Lie algebra $\mathfrak{g}$ is given by generators and relations we can choose a subset of the set of Lyndon words in the generators such that their standard bracketings form a basis of $\mathfrak{g}$. Together with the theorem of Poincaré, Birkhoff, and Witt (PBW) this provides a combinatorial description of the PBW basis of the enveloping algebra $U(\mathfrak{g})$.
Analogous PBW bases were found for deformed enveloping algebras by Yamane [48], Rosso [37] and Lusztig [27], whose proof is based on an action of the braid group. A different approach using Hall algebras to construct these PBW bases was found by Ringel [36]. For general (graded) algebras it is an interesting question as to whether they admit a PBW basis. In [20] Kharchenko proved a PBW result in the spirit of Lalonde and Ram for a class of pointed Hopf algebras which he calls character Hopf algebras. His proof uses combinatorial methods. The result can be reinterpreted in terms of braided Hopf algebras with diagonal braidings which are generated by a finite set of primitive elements.

In our main PBW theorem we give a generalization of Kharchenko's result to braided Hopf algebras of triangular type. The assumption that the braiding on the space of primitive generators is diagonal is replaced by the more general condition of triangularity. This seems to be the natural context for the existence of a PBW basis. The proof basically follows Kharchenko's approach, but the step from diagonal to triangular braidings requires new methods and ideas.

One application of our result leads to a generalization of Kharchenko's existence theorem for the PBW basis from [20] to Hopf algebras generated by an abelian group G and a finite-dimensional G-Yetter-Drinfeld module of skew-primitive elements; we do not require (as Kharchenko does) that the group action on the generators is given by characters. This is done with the help of Proposition 2.5.1 that allows us to lift PBW bases from the associated graded algebra of a filtered algebra to the filtered algebra itself.
As a second application of the PBW theorem we determine the structure of the Nichols algebras of low-dimensional $U_q(\mathfrak{sl}_2)$-modules, which was mostly unknown until now.

Triangular braidings are defined by a combinatorial property in 1.3.5. The main result of Chapter 3 is Theorem 3.3.6 that provides an alternative characterization of the triangularity property, leading to a better understanding of triangular braidings. As a tool we use a reduced version of the Faddeev-Reshetikhin-Takhtadzhyan Hopf algebra [9], which is inspired by a work of Radford [33]. We prove that triangular braidings are exactly the braidings coming from Yetter-Drinfeld modules over pointed Hopf algebras with abelian coradical which are completely reducible as modules over the coradical. This gives a conceptual characterization and emphasizes the close connection to pointed Hopf algebras with abelian coradical. Finally, we calculate the reduced FRT Hopf algebra for braidings induced by the quasi-$\mathcal{R}$-matrix of a deformed enveloping algebra, a result we will use in Chapter 4.

An important class of braided Hopf algebras generated by primitive elements is formed by Nichols algebras $\mathcal{B}(V,c)$ of braided vector spaces (V,c). The name refers to Nichols who studied them under the name of bialgebras of type one [31]. They are generalizations of the symmetric algebra $\mathcal{S}(V)$ of the vector space, where the flip map $\tau : V\otimes V \to V\otimes V, v\otimes w \mapsto w\otimes v$ is replaced by a braiding c. Nichols algebras play an important role in the classification program for finite-dimensional pointed Hopf algebras, which was started by Andruskiewitsch and Schneider [5]. Here and in the theory of quantum groups one is interested in the vector space dimension or the Gelfand-Kirillov dimension of these algebras and in representations by generators and relations. In general it turns out to be very hard to determine the structure of a Nichols algebra even for quite simple braidings. For braidings of diagonal type there are many results by Lusztig [27] (his algebra $\mathbf{f}$), Rosso [38] and Andruskiewitsch and Schneider. For more complicated braidings the knowledge is still very limited. While in the classification program one is mainly interested in braidings coming from Yetter-Drinfeld modules over groups, we consider the case of braidings induced by the quasi-$\mathcal{R}$-matrix of a deformed enveloping algebra. Since these braidings are triangular they can be seen as a special case of the situation described in Chapter 3. Apart from the case when the braiding is of Hecke type (see Example 1.4.9) almost nothing was known about Nichols algebras of triangular type .

In [1] Andruskiewitsch asks the following question: Given an integrable finite-dimensional $U_q(\mathfrak{g})$-module M with braiding c induced by the quasi-$\mathcal{R}$-matrix, what is the structure of the Nichols algebra $\mathcal{B}(M,c)$?
In Chapter 4 we answer his question with the help of a method that reduces the study of Nichols algebras of finite-dimensional $U_q(\mathfrak{g})$-modules M to the study of Nichols algebras of diagonal braidings. Actually we consider a much more general setting. Let H be a Hopf algebra with bijective antipode, V a Yetter-Drinfeld module over H and M a Yetter-Drinfeld module over $\mathcal{B}(V)\#H$. Theorem 4.3.1 states that under the assumption that the structure maps of M respect the natural grading of $\mathcal{B}(V)\#H$ there is an isomorphism

$$\mathcal{B}(M)\#\mathcal{B}(V)\simeq\mathcal{B}(M_H\oplus V),$$

where M_H is a Yetter-Drinfeld module over H associated to M. In the very special case where H is the group algebra of a free abelian group an analogous result is proved by Rosso in [38] using a different method.

To answer Andruskiewitsch's question, we use our characterization of triangular braidings from Chapter 3. In Section 3.4 we realized the braiding on a finite-dimensional $U_q(\mathfrak{g})$-module M induced by the quasi-$\mathcal{R}$-matrix as a Yetter-Drinfeld braiding over a pointed Hopf algebra U. It turns out that U decomposes into a biproduct $U = \mathcal{B}(V)\#kG$ for an abelian group G and from Theorem 4.3.1 we get $\mathcal{B}(M)\#\mathcal{B}(V)\simeq\mathcal{B}(M_{kG} \oplus V)$. Moreover V and $M_{kG} \oplus V$ have diagonal braidings, and we can apply known results about their Nichols algebras to obtain information on $\mathcal{B}(M)$.
The described method is used to prove a criterion for the finiteness of the Gelfand-Kirillov dimension of $\mathcal{B}(M)$. To the module M and its braiding we associate a matrix (b_{ij}) of rational numbers that is an extension of the Cartan matrix of $\mathfrak{g}$. Under some technical assumptions on the braiding, the Gelfand-Kirillov dimension of $\mathcal{B}(M)$ is finite if and only if (b_{ij}) is a Cartan matrix of finite type. For simple $\mathfrak{g}$ and simple modules M we give a complete list of all cases with finite Gelfand-Kirillov dimension (Table 4.1).
As an important second application of our method we describe explicitly the relations of $\mathcal{B}(M)$ under the assumption that the braided biproduct $\mathcal{B}(M)\#\mathcal{B}(V)$ is given by the quantum Serre relations (Remark 4.5.4). In particular this applies if $\mathcal{B}(M)$ has finite Gelfand-Kirillov dimension. Table 4.1 contains the degrees of the defining relations in the case that $\mathfrak{g}$ is simple, M is simple and the Gelfand-Kirillov dimension of $\mathcal{B}(M)$ is finite. All these relations were completely unknown (except for the very special case of braidings of Hecke type) before.
Due to missing information on Nichols algebras of diagonal braidings the results of both applications contain some technical restrictions.

The contents of Chapter 2 will appear in the Journal of Algebra [45]. I would like to thank all the people who helped to finish this thesis. First of all my advisor Prof. Dr. H.-J. Schneider for scientific guidance during the last three years. Also Priv.-Doz. Dr. Peter Schauenburg for a lot of useful hints. Then Gaston Garcia, Dr. István Heckenberger, Daniela Hobst, Birgit Huber and Tobias Stork for many interesting discussions and for proof-reading parts of the thesis and [45, 46]. Finally the State of Bavaria (Graduiertenförderung des bayerischen Staates) for a two-year scholarship. Last but not least special thanks to my parents and to my sister for their financial support and for being there for me.

Throughout this thesis we will mostly work over an arbitrary field k. Unadorned tensor products $\otimes$ are tensor products over k. Unless stated otherwise, all algebras are associative algebras over k with unit. For some results we need additional assumptions on k.

Chapter 1

Basic definitions

In this chapter we will mainly recall definitions and facts from the theory of Lie algebras and Hopf algebras. Most of the material is meant only as a quick reference for our notations and conventions. An important exception is Subsection 1.3.2, where the new notion of triangular braidings is introduced. This type of braiding is the central feature of braided Hopf algebras of triangular type as defined in Definition 1.4.14. For large parts of this chapter [19] is a good reference.

1.1 Lie algebras

In order to have the necessary notations fixed for the definition of quantum groups in Section 1.2.2 we will recall some facts on Lie algebras, especially on the classification of finite-dimensional semi-simple Lie algebras over the field of complex numbers. For more information and historical comments we suggest the books by Jacobson [14], Humphreys [13] and Kac [18].

1.1.1 Definition and the universal enveloping algebra

Definition 1.1.1. A *Lie algebra* is a pair $(\mathfrak{g}, [-,-])$, usually denoted by $\mathfrak{g}$, where $\mathfrak{g}$ is a vector space and

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

is a linear map (called the Lie bracket) satisfying

$$\begin{array}{lcl} \forall x \in \mathfrak{g} & [x,x] = 0 & \text{(antisymmetry)} \\ \forall x,y,z \in \mathfrak{g} & [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 & \text{(Jacobi identity).} \end{array}$$

A morphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras is a linear map such that for all $x, y \in \mathfrak{g} : \phi([x, y]) = [\phi(x), \phi(y)]$. A subspace $\mathfrak{a} \subset \mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if for all $a \in \mathfrak{a}, x \in \mathfrak{g}$ we have $[g, a] \in \mathfrak{a}$. An ideal of $\mathfrak{g}$ is called *simple*, if it has no proper sub-ideals. A Lie algebra is called *semi-simple* if it is the sum of its simple ideals.

Example 1.1.2. The space $\mathfrak{sl}_2 := ke \oplus kh \oplus kf$ is a Lie algebra with Lie bracket defined by the equations

$$[h, e] = 2e, \ \ [h, f] = -2f, \ \ [e, f] = h.$$

Example 1.1.3. For every associative algebra A we can form a Lie algebra A^- with underlying vector space A and Lie bracket defined by the commutator of the algebra A:

$$\forall x, y \in A : [x, y] := xy - yx.$$

This defines a functor $(-)^-$ from the category of associative algebras to the category of Lie algebras.

To a Lie algebra $\mathfrak{g}$ one associates an enveloping algebra $U(\mathfrak{g})$ in a natural way.

Definition 1.1.4. Let $\mathfrak{g}$ be a Lie algebra. An associative algebra U together with a morphism of Lie algebras $\iota : \mathfrak{g} \to U^-$ is called the *universal enveloping algebra* of $\mathfrak{g}$ if it satisfies the following universal property:
For every algebra A and every morphism of Lie algebras $\phi : \mathfrak{g} \to A^-$ there is a unique morphism of algebras $\psi : U \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi"] \arrow[dr, "\iota"'] & A^- \\
 & U^- \arrow[u, "\psi^-"']
\end{tikzcd}$$

For every Lie algebra $\mathfrak{g}$ there is a universal enveloping algebra $U(\mathfrak{g})$ and it is unique up to isomorphism. The following theorem due to Poincarè, Birkhoff, and Witt gives us a very useful description of the enveloping algebra.

Theorem 1.1.5. [13, Section 17.3. Corollary C] Let $\mathfrak{g}$ be a Lie algebra. Fix a k-linear basis S of $\mathfrak{g}$ and a total order $<$ on S. Then the set of all elements

$$\iota(s_1)^{e_1} \dots \iota(s_r)^{e_r}$$

with $r \in \mathbb{N}_0, s_1, \dots, s_r \in S, s_1 < \dots < s_r$ and $e_1, \dots, e_r \in \mathbb{N}$ form a k-linear basis of $U(\mathfrak{g})$.
In particular ι is injective and we can consider $\mathfrak{g}$ as a Lie subalgebra of $U(\mathfrak{g})^-$.

Remark 1.1.6. Assume $\operatorname{char} k = p > 0$ and let $\mathfrak{g}$ be a restricted Lie algebra of characteristic p (see e.g. [14, V.7]). In this case one can define a restricted universal enveloping algebra $u(\mathfrak{g})$ and we obtain a similar theorem:
Again fix a k-linear basis S of $\mathfrak{g}$ and a total order $<$ on S. Then the set of all elements

$$\iota(s_1)^{e_1} \dots \iota(s_r)^{e_r}$$

with $r \in \mathbb{N}_0, s_1, \dots, s_r \in S, s_1 < \dots < s_r$ and $1 \leq e_1, \dots, e_r < p$ form a k-linear basis of $u(\mathfrak{g})$.

These two theorems are the prototypes for the PBW Theorem 2.2.4 for braided Hopf algebras of triangular type that we will prove in Chapter 2.

1.1.2 Root systems and Dynkin diagrams

The classification of complex finite-dimensional semi-simple Lie algebras describes these objects in terms of root systems and their Dynkin diagrams. In this section we will recall the necessary definitions and facts. The main reference is the book by Humphreys [13].

Definition 1.1.7. A *root system* is a pair (V, Φ), where V is a euclidean vector space with scalar product $(-,-)$ and Φ is a subset of V satisfying

(R1) Φ is finite, spans V and does not contain 0,

(R2) $\forall \alpha \in \Phi : \mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$,

(R3) $\forall \alpha, \beta \in \Phi : \beta - \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi$ and

(R4) $\forall \alpha, \beta \in \Phi : \langle \alpha, \beta \rangle := \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

An isomorphism $\phi : (V, \Phi) \to (V', \Phi')$ of root systems is a linear isomorphism $\phi : V \to V'$ that maps Φ into Φ' and satisfies $\langle \alpha, \beta \rangle = \langle \phi(\alpha), \phi(\beta) \rangle$ for all $\alpha, \beta \in \Phi$. Note that replacing the scalar product by a real multiple we obtain an isomorphic root system.

For a root system $R = (V, \Phi)$ the group $\mathbb{Z}\Phi \subset V$ is called the *root lattice* of R and the group

$$\begin{aligned}\Lambda &:= \{\lambda \in V | \forall \alpha \in \Phi : \langle \alpha, \lambda \rangle \in \mathbb{Z}\} \\ &= \{\lambda \in V | \forall \alpha \in \Phi : (\alpha, \lambda) \in \frac{(\alpha, \alpha)}{2}\mathbb{Z}\}\end{aligned}$$

is called the *weight lattice* of R.
A subset $\Pi \subset \Phi$ is called a *basis of the root system* R if Π is a basis of V and every $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha$ with integers k_α that are all non-positive or all non-negative. Every root system R has a basis Π and we define the *Cartan matrix* $(a_{\alpha\beta})_{\alpha,\beta \in \Pi}$ of R (with respect to the basis Π) by

$$a_{\alpha,\beta} := \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \ \forall \alpha, \beta \in \Pi.$$

The Cartan matrix is well-defined up to a permutation of the index set and in this sense does not depend on the choice of the basis Π.
To have a nice picture of the root systems we associate a *Dynkin diagram* to their Cartan matrix in the following way. The Dynkin diagram is an oriented graph with vertex set Π (a fixed basis of the root system). For $\alpha, \beta \in \Pi$ we draw $a_{\beta\alpha} a_{\alpha\beta}$ lines between α and β. If $|a_{\alpha\beta}| > 1$ we draw an arrow tip pointing towards α.

Remark 1.1.8. The Dynkin diagrams of root systems are unions of finitely many of the connected Dynkin diagrams in Table 1.1. This is a key result in the classification of finite-dimensional semi-simple complex Lie algebras. A proof can be found in [14] for example.

After choosing a basis we can define a partial order on the root lattice.

Definition 1.1.9. Assume we are given a root system (V, Φ) and a fixed basis Π of it. An element $\mu \in \mathbb{Z}\Phi$ of the root lattice will be called *positive* if it is a non-zero linear combination of basis elements with non-negative coefficients. It will be called *negative* if it is a non-zero linear combination of basis elements with non-positive coefficients. Let Φ^+ resp. Φ^- denote the set of positive resp. negative roots. We define a partial order on $\mathbb{Z}\Phi$ by

$$\mu > \nu \ :\Leftrightarrow \ \mu - \nu \text{ is positive}$$

for all $\mu, \nu \in \mathbb{Z}\Phi$.

For a positive root

$$\mu = \sum_{\alpha \in \Pi} k_\alpha \alpha$$

define the *height* of μ by $\operatorname{ht} \mu := \sum_{\alpha \in \Pi} k_\alpha$.

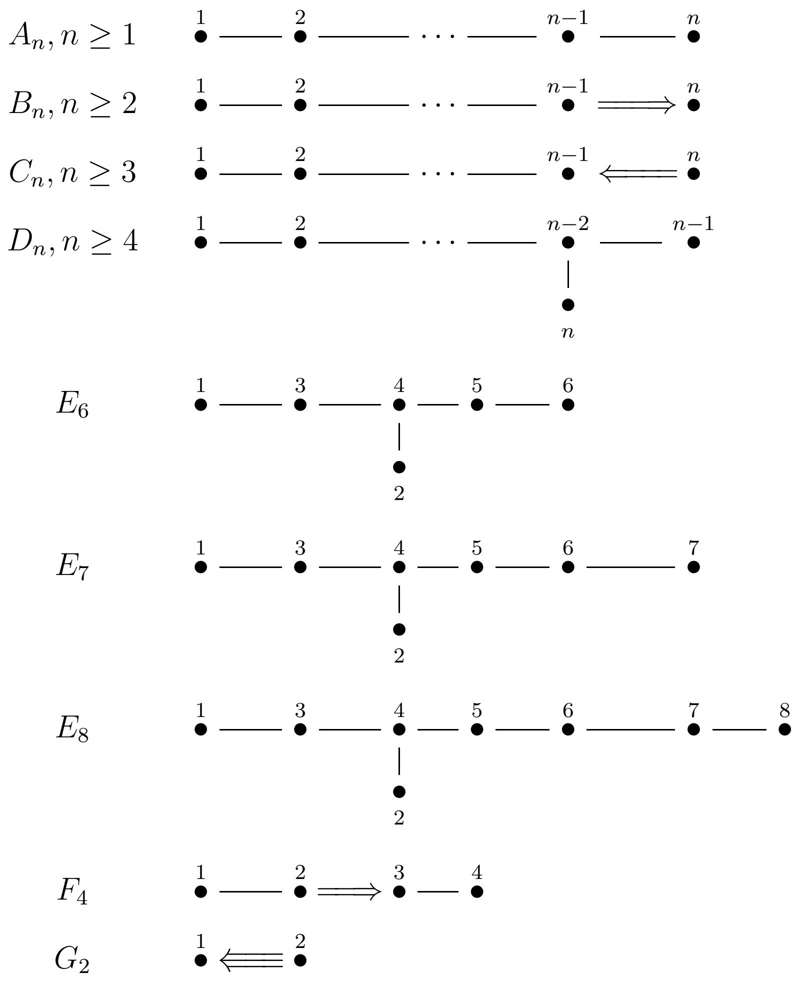


Table 1.1: Connected Dynkin diagrams of root systems

1.1.3 The classification of semi-simple Lie algebras

The following theorem contains the classification of all finite-dimensional semi-simple Lie algebras over the complex numbers. This problem was solved by Killing and Cartan at the end of the 19th century, motivated by the classification of space forms (Raumformen); see [11] for a historical overview. The presentation by generators and relations given here is due to Serre.

Theorem 1.1.10. The following construction yields a one-to-one correspondence between complex finite-dimensional semi-simple Lie algebras (up to isomorphism) and root systems (up to isomorphism).
Let $A = (a_{\alpha\beta})_{\alpha,\beta\in\Pi}$ be the Cartan matrix of a root system R. Then the complex Lie algebra $\mathfrak{g}_R$ generated by $e_\alpha, h_\alpha, f_\alpha, \alpha \in \Pi$ with relations

$$\begin{aligned} [h_\alpha, h_\beta] &= 0, \\ [h_\alpha, e_\beta] &= a_{\alpha\beta} e_\beta, \\ [h_\alpha, f_\beta] &= -a_{\alpha\beta} f_\beta, \\ [e_\alpha, f_\beta] &= \delta_{\alpha,\beta} h_\alpha \end{aligned}$$

for all $\alpha, \beta \in \Pi$ and the Serre relations

$$\begin{aligned} \forall \alpha \neq \beta \in \Pi : \quad & \mathrm{ad}(e_\alpha)^{1-a_{\alpha\beta}}(e_\beta) = 0 \text{ and} \\ \forall \alpha \neq \beta \in \Pi : \quad & \mathrm{ad}(f_\alpha)^{1-a_{\alpha\beta}}(f_\beta) = 0. \end{aligned}$$

is finite-dimensional and semi-simple.
$\mathfrak{g}_R$ is simple (i.e. it has no proper ideals) if and only if the Dynkin diagram of R is connected.

There is a similar characterization of affine Lie algebras; we refer to Kac [18] for details. As the semi-simple Lie algebras are constructed only using the Cartan matrix of the root system we see that the root system is uniquely determined by its Cartan matrix up to isomorphism. Furthermore we can calculate the Cartan matrix $(a_{\alpha\beta})_{\alpha,\beta\in\Pi}$ of R from its Dynkin diagram, because in the proof it turns out that for all $\alpha \neq \beta \in \Pi$ we have $a_{\alpha\alpha} = 2$, $a_{\alpha\beta} \in \{0, -1, -2, -3\}$ and $a_{\alpha\beta} = 0$ if and only if $a_{\beta\alpha} = 0$. For example the Cartan matrix of G_2 is

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

1.2 Coalgebras, bialgebras and Hopf algebras

In this section we recall the definitions of bialgebras and Hopf algebras, which form the basic structures for our work. There are many textbooks on this subject. Our main references are the books by Sweedler [43] and Montgomery [30].

1.2.1 Coalgebras

An important ingredient in the definition of a Hopf algebra is the notion of a coalgebra, which is dual to that of an algebra.

Definition 1.2.1. A *coalgebra* is a vector space C together with two linear maps

$$\Delta : C \to C \otimes C \text{ and } \varepsilon : C \to k$$

called the *comultiplication* resp. the *counit* that satisfy

$$\begin{aligned} (\Delta \otimes \mathrm{id}_C)\Delta &= (\mathrm{id}_C \otimes \Delta)\Delta \qquad &\text{(coassociativity) and}\\ (\varepsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C &= (\mathrm{id}_C \otimes \varepsilon)\Delta \qquad &\text{(counitality).} \end{aligned}$$

A morphism $\phi : C \to D$ of coalgebras is a linear map such that

$$(\phi \otimes \phi)\Delta_C = \Delta_D \phi \text{ and } \varepsilon_D \phi = \varepsilon_C.$$

In order to be able to perform calculations in coalgebras we use a common version of the Heyneman-Sweedler notation for the comultiplication. If C is a coalgebra and $c \in C$ we write formally

$$\Delta(c) = c_{(1)} \otimes c_{(2)},$$

always keeping in mind that $c_{(1)} \otimes c_{(2)}$ is in general not a simple tensor. The coassociativity axiom allows us to write for higher "powers" of the comultiplication

$$\Delta_n(c) := (\Delta \otimes \mathrm{id}_{V^{\otimes n-2}}) \dots (\Delta \otimes \mathrm{id}_V)\Delta = c_{(1)} \otimes \dots \otimes c_{(n)}.$$

Let C be a coalgebra. An element $g \in C$ is called *group-like* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Let $G(C)$ denote the set of group-like elements of C. It is always linearly independent. For two group-likes $g, h \in G(C)$ an element $x \in C$ is *g, h-skew primitive* if $\Delta(x) = x \otimes g + h \otimes x$ and denote the space of g, h-skew primitive elements by $P_{g,h}(C)$.
A coalgebra is *simple* if it has no nontrivial subcoalgebras. It is said to be *pointed* if every simple subcoalgebra is one-dimensional, i.e. spanned by a

group-like element. For a coalgebra C define the *coradical* $\operatorname{Corad} C$ or C_0 of C as the sum of all simple subcoalgebras. The coalgebra is called *irreducible* if $\operatorname{Corad} C$ is a simple coalgebra and it is called *connected* if the coradical is one-dimensional.

A useful tool in the theory of coalgebras are filtrations. A coalgebra filtration $(C_n)_{n\geq 0}$ of C is a filtration of the vector space C such that for all $n \geq 0$

$$\Delta(C_n) \subset \sum_{i+j=n} C_i \otimes C_j.$$

Every coalgebra has an important filtration, the coradical filtration defined by the wedge product (see [43, Chapter IX]):

$$C_n := \wedge^n C_0,$$

where $C_0 = \operatorname{Corad} C$. More generally it is true that the lowest term of a coalgebra filtration always contains the coradical. A graded coalgebra C is a coalgebra equipped with a vector space decomposition $C = \oplus_{n\geq 0} C(n)$ such that for all $n \geq 0$

$$\Delta(C(n)) \subset \bigoplus_{i+j=n} C(i) \otimes C(j)$$

and $\varepsilon | C(n) = 0$ for all $n \geq 1$. For every filtered coalgebra one can construct an associated graded coalgebra $\operatorname{gr} C$ by setting $\operatorname{gr} C(n) := C_n / C_{n-1}$ for $n \geq 0$ (with $C_{-1} := \{0\}$ as usual) and defining the comultiplication and counit in a natural way.

As coalgebras are dual to algebras we are also interested in the dual of a module, a comodule over a coalgebra.

Definition 1.2.2. Let C be a coalgebra. A (left) *comodule* over C is a pair (M, δ), or just M, consisting of a vector space M and a linear map

$$\delta : M \to C \otimes M$$

called the *coaction* satisfying

$$\begin{aligned} (\Delta \otimes \mathrm{id}_M)\delta &= (\mathrm{id}_C \otimes \delta)\delta && \text{(coassociativity) and} \\ (\varepsilon \otimes \mathrm{id}_M)\delta &= \mathrm{id}_M && \text{(counitality).} \end{aligned}$$

A morphism $f : M \to M'$ of comodules is a k-linear map such that

$$\delta_{M'} f = (\mathrm{id}_C \otimes f)\delta_M.$$

Such a map is also called a *colinear* map.

Similarly to the comultiplication we use a version of the Heyneman-Sweedler notation for coactions:

$$\delta_M(m) := m_{(-1)} \otimes m_{(0)}.$$

The coassociativity axiom then reads

$$(m_{(-1)})_{(1)} \otimes (m_{(-1)})_{(2)} \otimes m_{(0)} = m_{(-1)} \otimes (m_{(0)})_{(-1)} \otimes (m_{(0)})_{(0)}.$$

This expression is written as $m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)}$.

1.2.2 Bialgebras and Hopf algebras

Let (C, Δ, ε) be a coalgebra and A be an algebra with multiplication map $\nabla : A \otimes A \to A$ and unit map $\eta : k \to A$. Then the space $\mathrm{Hom}_k(C, A)$ becomes an algebra with multiplication given by the *convolution product*

$$\forall f, g \in \mathrm{Hom}_k(C, A) : f \star g := \nabla(f \otimes g)\Delta$$

and unit $\eta\varepsilon$.
Note that for an algebra A resp. a coalgebra C, $A \otimes A$ resp. $C \otimes C$ is again an algebra resp. a coalgebra by

$$\begin{aligned} (a \otimes b)(a' \otimes b') &:= aa' \otimes bb' \text{ and} \\ \Delta(c \otimes d) &:= \left(c_{(1)} \otimes d_{(1)}\right) \otimes \left(c_{(2)} \otimes d_{(2)}\right). \end{aligned}$$

Definition 1.2.3. A *bialgebra* is a quintuple $(H, \nabla, \eta, \Delta, \varepsilon)$, where (H, ∇, η) is an algebra, (H, Δ, ε) is a coalgebra and one of the following equivalent conditions is satisfied:

- $\Delta : H \to H \otimes H$, $\varepsilon : H \to k$ are algebra morphisms.
- $\nabla : H \otimes H \to H$, $\eta : k \to H$ are coalgebra morphisms.

The bialgebra H is called a *Hopf algebra* if the identity map id_H is invertible in the convolution algebra $\mathrm{End}_k(H)$, i.e. there is a map $\mathbf{S} \in \mathrm{End}_k(H)$ (called the *antipode* of H) such that for all $h \in H$

$$h_{(1)}\mathbf{S}(h_{(2)}) = \varepsilon(h)1 = \mathbf{S}(h_{(1)})h_{(2)}.$$

A morphism $\phi : H \to H'$ of bialgebras is a morphism of algebras and coalgebras.

A bialgebra is called pointed, connected resp. irreducible if the underlying coalgebra has this property. In a bialgebra we have a distinguished group-like element, the unit 1. The $1,1$-skew primitive elements are also called *primitive elements*. The space of primitive elements of H is denoted by $P(H)$.

Example 1.2.4. Let G be a group. Then the group algebra kG becomes a (pointed) Hopf algebra with

$$\Delta(g) = g \otimes g, \varepsilon(g) = 1, \mathbf{S}(g) = g^{-1}$$

for all $g \in G$.

Example 1.2.5. Let $\mathfrak{g}$ be a Lie algebra. Then the universal enveloping algebra $U(\mathfrak{g})$ is a connected Hopf algebra with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \varepsilon(x) = 0, \mathbf{S}(x) = -x$$

for all $x \in \mathfrak{g}$. Thus the elements of $\mathfrak{g}$ are primitive elements. A theorem by Friedrich [14, V.4.] states that if $\operatorname{char} k = 0$ we have $P(U(\mathfrak{g})) = \mathfrak{g}$; Jacobson only states the theorem for free Lie algebras, but it is true for arbitrary $\mathfrak{g}$. Assume $\operatorname{char} k = p > 0$ and let $\mathfrak{g}$ be a restricted Lie algebra of characteristic p. The restricted enveloping algebra $u(\mathfrak{g})$ is a Hopf algebra with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \varepsilon(x) = 0, \mathbf{S}(x) = -x$$

for all $x \in \mathfrak{g}$. In this case we have $P(u(\mathfrak{g})) = \mathfrak{g}$.

In Hopf algebras we have a generalization of the adjoint action known from groups and Lie algebras. For a Hopf algebra H define the *adjoint action* of an element $h \in H$ by

$$\operatorname{ad}(h) : H \to H, \;\; x \mapsto h_{(1)} x \mathbf{S}(h_{(2)}).$$

1.2.3 Deformed enveloping algebras

In the 1980's new and very interesting examples of non-commutative and non-cocommutative Hopf algebras were found, starting with papers by Kulish and Reshitikin [23], Drinfeld [8] and Jimbo [16]. These developed to a whole new class of Hopf algebras, the deformed enveloping algebras of semi-simple Lie algebras, usually also called quantum groups [4]. We will recall the necessary definitions and fix some notations. Our main reference are the books of Jantzen [15] and Lusztig [27].
Assume that $\operatorname{char} k = 0$ and $q \in k$ is not a root of unity. Furthermore let $\mathfrak{g}$ be a complex finite-dimensional semi-simple Lie algebra with root system

(V, Φ), basis of the root system Π and Cartan matrix $(a_{\alpha\beta})_{\alpha,\beta\in\Pi}$. We will use the same normalization for the scalar product of the root system as Jantzen [15, 4.1]. In this setting we have $(\alpha, \alpha) \in \{2, 4, 6\}$ and $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Pi$. For $\alpha \in \Pi$ let $d_\alpha := \frac{(\alpha,\alpha)}{2}$ and $q_\alpha := q^{d_\alpha}$. For all $\alpha, \beta \in \Pi$ we have $(\alpha, \beta) = d_\alpha a_{\alpha\beta}$.

Definition 1.2.6. The *quantum enveloping algebra* $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is the algebra generated by the symbols $E_\alpha, K_\alpha, K_\alpha^{-1}, F_\alpha, \alpha \in \Pi$ subject to the relations

$$\begin{aligned} K_\alpha K_\alpha^{-1} = \;& 1 = K_\alpha^{-1} K_\alpha, \\ K_\alpha K_\beta \;&=\; K_\beta K_\alpha, \\ K_\alpha E_\beta K_\alpha^{-1} \;&=\; q^{(\alpha,\beta)} E_\beta, \\ K_\alpha F_\beta K_\alpha^{-1} \;&=\; q^{-(\alpha,\beta)} F_\beta, \\ E_\alpha F_\beta - F_\beta E_\alpha \;&=\; \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \end{aligned}$$

for all $\alpha, \beta \in \Pi$ and the quantum Serre relations for all $\alpha \neq \beta \in \Pi$:

$$\begin{aligned} \mathrm{ad}(E_\alpha)^{1-a_{\alpha\beta}}(E_\beta) = 0, \\ \mathrm{ad}(F_\alpha)^{1-a_{\alpha\beta}}(F_\beta) = 0. \end{aligned}$$

Note that the Serre relations only make sense if we define a Hopf algebra structure on the algebra generated by $E_\alpha, K_\alpha, K_\alpha^{-1}, F_\alpha, \alpha \in \Pi$ subject to the first set of relations. This can be done in the same way as for $U_q(\mathfrak{g})$ in the following proposition.

Proposition 1.2.7. $U_q(\mathfrak{g})$ becomes a Hopf algebra with

$$\begin{aligned} \Delta(K_\alpha) = K_\alpha \otimes K_\alpha, &\quad \varepsilon(K_\alpha) = 1, \quad \mathbf{S}(K_\alpha) = K_\alpha^{-1}, \\ \Delta(E_\alpha) = K_\alpha \otimes E_\alpha + E_\alpha \otimes 1, &\quad \varepsilon(E_\alpha) = 0, \quad \mathbf{S}(E_\alpha) = -K_\alpha^{-1} E_\alpha, \\ \Delta(F_\alpha) = 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}, &\quad \varepsilon(F_\alpha) = 0, \quad \mathbf{S}(F_\alpha) = -F_\alpha K_\alpha. \end{aligned}$$

Proof. See e.g. Jantzen's book [15]. □

For all $\mu \in \mathbb{Z}\Phi$ define the *root space*

$$U_\mu := \{u \in U_q(\mathfrak{g}) | \forall \alpha \in \Phi : K_\alpha u K_\alpha^{-1} = q^{(\alpha,\mu)} u\}.$$

Then we have

$$U_q(\mathfrak{g}) = \bigoplus_{\mu \in \mathbb{Z}\Phi} U_\mu.$$

There are several important subalgebras of $U_q(\mathfrak{g})$.

Definition 1.2.8. We have the Hopf subalgebra $U_q^{\geq 0}(\mathfrak{g})$ generated by the K_α, K_α^{-1} and the $E_\alpha, \alpha \in \Pi$ and the Hopf subalgebra $U_q^{\leq 0}(\mathfrak{g})$ generated by the K_α, K_α^{-1} and the $F_\alpha, \alpha \in \Pi$.
The subalgebra generated by the $E_\alpha, \alpha \in \Pi$ is called the *positive part* of $U_q(\mathfrak{g})$ and is denoted by $U_q^+(\mathfrak{g})$. Similarly the subalgebra generated by the $F_\alpha, \alpha \in \Pi$ is called the *negative part* of $U_q(\mathfrak{g})$ and is denoted by $U_q^-(\mathfrak{g})$.

These positive and the negative parts are not Hopf subalgebras because they are not subcoalgebras. In fact they are braided Hopf algebras, as we will see in Example 1.4.10. Each of these subalgebras has a similar root space decomposition as $U_q(\mathfrak{g})$ and the root spaces are denoted by $U_\mu^{\geq 0}, U_\mu^{\leq 0}, U_\mu^+$ and U_μ^- respectively.
Now we will review some facts on integrable $U_q(\mathfrak{g})$-modules.

Definition 1.2.9. For a $U_q(\mathfrak{g})$-module M and an element of the weight lattice $\lambda \in \Lambda$ define the *weight space*

$$M_\lambda := \left\{ m \in M | \forall \alpha \in \Pi : K_\alpha m = q^{(\alpha,\lambda)} m \right\}.$$

M is called *integrable* if for each $m \in M$ and $\alpha \in \Pi$ there is $n \in \mathbb{N}$ such that $E_\alpha^n m = F_\alpha^n m = 0$ and M is the direct sum of its weight spaces

$$M = \bigoplus_{\lambda \in \Lambda} M_\lambda.$$

M is called a *module of highest weight* λ if there is $0 \neq m \in M_\lambda$ such that $E_\alpha m = 0$ for all $\alpha \in \Pi$ and m generates M as a $U_q(\mathfrak{g})$-module.
A weight $\lambda \in \Lambda$ is called a *dominant weight* if for all $\alpha \in \Pi$ we have $(\lambda, \alpha) \geq 0$. The set of dominant weights is denoted by Λ^+.

We need the following theorem on the structure of integrable modules.

Theorem 1.2.10. [27, Corollary 6.2.3. and Proposition 6.3.6.]

- For every dominant weight $\lambda \in \Lambda^+$ there is a simple $U_q(\mathfrak{g})$-module $M(\lambda)$ of highest weight λ. It is finite-dimensional and unique up to isomorphism.
- Every integrable $U_q(\mathfrak{g})$-module is a direct sum of simple $U_q(\mathfrak{g})$-modules of the form $M(\lambda)$ with $\lambda \in \Lambda^+$.

Finally, a very important feature of the deformed enveloping algebras is the quasi-$\mathcal{R}$-matrix that allows to define braidings on integrable modules.

Remark 1.2.11. There is an interesting non-degenerate bilinear form between the positive and the negative part

$$(-,-) : U_q^-(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) \to k,$$

which is in some sense compatible with the algebra and coalgebra structure. For further information refer to [15, Chapter 6].
For $u \in U_{-\mu}^-, v \in U_\nu^+$ we have $(u,v) = 0$ whenever $\mu \neq \nu$. Thus for all $\mu \geq 0$ the restriction of $(-,-)$ to $U_{-\mu}^- \otimes U_\mu^+$ is non-degenerate. Fix a basis $u_1^\mu, \dots, u_{r_\mu}^\mu$ of U_μ^+, let $v_1^\mu, \dots, v_{r_\mu}^\mu$ be the dual basis of $U_{-\mu}^-$ and define

$$\Theta_\mu := \sum_{i=1}^{r_\mu} v_i^\mu \otimes u_i^\mu \in U_q^-(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).$$

The (formal) sum

$$\Theta := \sum_{\mu \geq 0} \Theta_\mu$$

is called the *quasi-$\mathcal{R}$-matrix* of $U_q(\mathfrak{g})$. For all $\mu \geq 0$ we write formally

$$\Theta_\mu =: \Theta_\mu^- \otimes \Theta_\mu^+,$$

always keeping in mind that this is in general not a decomposeable tensor.

1.3 Yetter-Drinfeld modules and braidings

In this section we fix notations and definitions concerning braidings on finite-dimensional vector spaces. Braidings are an important tool in the construction of invariants of knots and links. Apart from this, braided structures appear naturally in the theory of Hopf algebras.
Furthermore we introduce triangular braidings, which are closely connected braided Hopf algebras of triangular type.

1.3.1 Yetter-Drinfeld modules

Definition 1.3.1. Let H be a bialgebra. A (left-left) *Yetter-Drinfeld module* M over H is a left H-module and left H-comodule such that the following compatibility condition holds for all $h \in H, m \in M$

$$(h_{(1)}m)_{(-1)}h_{(2)} \otimes (h_{(1)}m)_{(0)} = h_{(1)}m_{(-1)} \otimes h_{(2)}m_{(0)}.$$

The category of Yetter-Drinfeld modules over H with morphisms given by linear and colinear maps is denoted by ${}_H^H\mathcal{YD}$.

If H is a Hopf algebra, the condition above is equivalent to the property that for all $h \in H, m \in M$

$$\delta(hm) = h_{(1)} m_{(-1)} \mathbf{S}(h_{(3)}) \otimes h_{(2)} m_{(0)}.$$

The tensor product of two Yetter-Drinfeld modules M, N is again a Yetter-Drinfeld module with structure

$$h(m \otimes n) := h_{(1)} m \otimes h_{(2)} n \text{ and } \delta(m \otimes n) = m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}$$

for all $h \in H, m \in M, n \in N$. If H is a Hopf algebra with bijective antipode, also the dual of a finite-dimensional Yetter-Drinfeld module M is again a Yetter-Drinfeld module by

$$(h\varphi)(m) := \varphi(\mathbf{S}(h)m), \delta(\varphi) := \sum_{i=1}^{r} \mathbf{S}^{-1}(m_{i_{(-1)}}) \otimes \varphi(m_{i_{(0)}}) m^i$$

for $h \in H, m \in M, \varphi \in M^*$. Here $(m_i)_{1 \leq i \leq r}$ is a basis of M with dual basis $(m^i)_{1 \leq i \leq r}$. This turns the category of finite-dimensional Yetter-Drinfeld modules ${}_H^H\mathcal{YD}^{fd}$ into a rigid monoidal category.
For every pair of Yetter-Drinfeld modules $M, N \in {}_H^H\mathcal{YD}$ we can define an homomorphism called the *braiding*:

$$c_{M,N} : M \otimes N \to N \otimes M, \;\; c(m \otimes n) := m_{(-1)} n \otimes m_{(0)}.$$

This homomorphism is natural in M and N. On every triple M, N, P of Yetter-Drinfeld modules it satisfies the *braid equation*

$$\begin{aligned}(c_{N,P} \otimes \mathrm{id}_M)(\mathrm{id}_N \otimes c_{M,P})(c_{M,N} \otimes \mathrm{id}_P) = \\ = (\mathrm{id}_P \otimes c_{M,N})(c_{M,P} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes c_{N,P}).\end{aligned}$$

If H is a Hopf algebra with bijective antipode, then the $c_{M,N}$ are in fact isomorphisms and the category ${}_H^H\mathcal{YD}^{fd}$ is a rigid braided monoidal category with these braidings. For further information on braided monoidal categories we suggest [17] or Kassel's book [19].

1.3.2 Braidings

An important special case of the braid equation is obtained if only one module is considered. This leads to the notion of a braided vector space. We view the braiding as a generalization of the usual flip map

$$\tau : V \otimes V \to V \otimes V, \;\; \tau(v \otimes w) = w \otimes v.$$

Replacing the flip map with a braiding is the basic idea for the definition of braided bialgebras as generalizations of usual bialgebras. In this sense Nichols algebras (see Definition 1.4.7) can be seen as generalizations of symmetric algebras.

Definition 1.3.2. A *braided vector space* is a pair (V, c), where V is a vector space and $c \in \mathrm{Aut}(V \otimes V)$ is a linear automorphism that satisfies the braid equation

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c).$$

We say that c is a braiding on V. A morphism $\phi : (V, c) \to (V', c')$ of braided vector spaces is a linear map $\phi : V \to V'$ such that

$$(\phi \otimes \phi)c = c'(\phi \otimes \phi).$$

As a braiding is meant to be a generalization of the usual flip map τ we define further isomorphisms

$$c_{n,m} : V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes m} \otimes V^{\otimes n}$$

(that "interchange" $V^{\otimes m}$ and $V^{\otimes n}$) inductively by

$$c_{0,0} := \mathrm{id}_k, c_{1,0} := \mathrm{id}_V =: c_{0,1},$$

$$c_{1,m+1} := (\mathrm{id}_V \otimes c_{1,m})(c_{1,1} \otimes \mathrm{id}_{V^{\otimes m}}) \text{ and}$$

$$c_{n+1,m} := (c_{n,m} \otimes \mathrm{id}_V)(\mathrm{id}_{V^{\otimes n}} \otimes c_{1,m}).$$

Usually one restricts to certain classes of braidings. *Braidings of diagonal type* form the simplest (though very interesting) class of braidings. A braiding c on a vector space V is called *of diagonal type* (with respect to the basis X) if there are a basis $X \subset V$ of V and for all $x, y \in X$ coefficients $q_{xy} \in k^\times$ such that

$$c(x \otimes y) = q_{xy} y \otimes x.$$

The most common example is the usual flip map. In the theory of quantum groups one usually deals with braidings of Cartan type resp. of Frobenius-Lusztig type. To define this we need the notion of a generalized Cartan matrix.

Definition 1.3.3. [18] Let X be a finite set. A *generalized Cartan matrix* (with index set X) is a matrix $(a_{xy})_{x,y\in X}$ with integer entries satisfying

- $\forall x \in X : a_{xx} = 2$,
- $\forall x, y \in X, x \neq y : a_{xy} \leq 0$ and
- $\forall x, y \in X : a_{xy} = 0 \Rightarrow a_{yx} = 0$.

Cartan matrices of root systems are generalized Cartan matrices as in this definition. A generalized Cartan matrix is called a *Cartan matrix of finite type* if it is the Cartan matrix of a root system.

Following [5] we say that a braiding is of *Cartan type* if it is of diagonal type with respect to a basis X and there is a generalized Cartan matrix $(a_{xy})_{x,y\in X}$ such that the coefficients q_{xy} of the braiding satisfy

$$\forall x, y \in X : q_{xy}q_{yx} = q_{xx}^{a_{xy}}.$$

A braiding is of *Frobenius-Lusztig type* (FL-type) if it is of diagonal type and there are a scalar $q \in k^\times$, a generalized Cartan matrix $(a_{xy})_{x,y\in X}$ and relatively prime positive integers $(d_x)_{x\in X}$ such that for all $x, y \in X$ we have

$$d_x a_{xy} = d_y a_{yx},$$

and the coefficients of the braiding are

$$q_{xy} = q^{d_x a_{xy}}.$$

Remark 1.3.4. If k is algebraically closed, the braidings of diagonal type are exactly those braidings that arise from Yetter-Drinfeld modules over abelian groups which are completely reducible as modules.

This thesis deals with a generalization of braidings of diagonal type, called triangular braidings. These will be defined now by a combinatorial property which is the natural context for the proof of the PBW Theorem 2.2.4. A description similar to that in Remark 1.3.4 will be obtained in Chapter 3.

Definition 1.3.5. Let V be a vector space with a totally ordered basis X and $c \in \mathrm{End}(V \otimes V)$.
The endomorphism c will be called *left triangular* (with respect to the basis X) if for all $x, y, z \in X$ with $z>y$ there exist $\gamma_{x,y} \in k$ and $v_{x,y,z} \in V$ such that for all $x, y \in X$

$$c(x \otimes y) = \gamma_{x,y} y \otimes x + \sum_{z>y} z \otimes v_{x,y,z}.$$

The endomorphism c will be called *right triangular* (with respect to the basis X) if for all $x, y, z \in X$ with $z>x$ there exist $\beta_{x,y} \in k$ and $w_{x,y,z} \in V$ such that for all $x, y \in X$

$$c(x \otimes y) = \beta_{x,y} y \otimes x + \sum_{z>x} w_{x,y,z} \otimes z.$$

A braided vector space (V, c) will be called left (resp. right) triangular with respect to the basis X if c is left (resp. right) triangular with respect to the basis X.

Assume that the braided vector space (V, c) is left triangular and adopt the notation from the definition. Then the map $d : V \otimes V \to V \otimes V$ defined by

$$d(x \otimes y) = \gamma_{x,y} y \otimes x$$

for all $x, y \in X$ is a braiding of diagonal type on V and it is called the *diagonal component* of c. It will be an important tool in the proof of the PBW theorem in Chapter 2. Similarly we define the diagonal component for right triangular braidings.

Remark 1.3.6. The name "left triangular" is motivated by the following observation: Assume in the situation of the definition that V has dimension n and denote by $B = (b_1, \dots, b_{n^2})$ the basis $\{x \otimes y | x, y \in X\}$ of $V \otimes V$ ordered lexicographically. By $B^{op} = (b'_1, \dots, b'_{n^2})$ denote the basis obtained from B by flipping the sides of every tensor (not changing the order). Then the matrix $A \in \mathrm{GL}(n^2, k)$ satisfying $c(b'_1, \dots, b'_{n^2}) = (b_1, \dots, b_{n^2})A$ has the following form:

$$A = \begin{pmatrix} D_1 & 0 & \dots & 0 \\ \star & D_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \dots & \star & D_n \end{pmatrix},$$

where $D_1, \dots, D_n \in \mathrm{GL}(n, k)$ are diagonal matrices. If the braiding were of diagonal type, this matrix would be diagonal.

Example 1.3.7. Assume $\operatorname{char} k = 0$ and let $0 \neq q \in k$ be not a root of unity. Let $\mathfrak{g}$ be a semi-simple finite-dimensional complex Lie algebra with root system (V, Φ), weight lattice Λ and let Π be a basis of the root system. Every integrable $U_q(\mathfrak{g})$-module is equipped with a class of braidings coming from the quasi-$\mathcal{R}$-matrix of $U_q(\mathfrak{g})$ (see also [15, Chapter 6]). To define these fix a function $f : \Lambda \times \Lambda \to k^\times$ that satisfies

$$f(\lambda + \nu, \mu) = q^{-(\nu,\mu)} f(\lambda, \mu) \text{ and } f(\lambda, \mu + \nu) = q^{-(\lambda,\nu)} f(\lambda, \mu) \qquad (1.1)$$

for all $\lambda, \mu \in \Lambda, \nu \in \mathbb{Z}\Phi$. For $\mu \geq 0$ let Θ_μ be the corresponding component of the quasi-$\mathcal{R}$-matrix as defined in 1.2.11. For each pair of integrable $U_q(\mathfrak{g})$-modules M, N there is a $U_q(\mathfrak{g})$-linear isomorphism

$$c^f_{M,N} : M \otimes N \to N \otimes M$$

such that for $m \in M_\lambda, n \in N_{\lambda'}$ we have

$$c^f_{M,N}(m \otimes n) = f(\lambda', \lambda) \sum_{\mu \geq 0} \Theta_\mu(n \otimes m).$$

Note that by Theorem 1.2.10 this expression is well-defined because the sum is actually finite. On every triplet of integrable $U_q(\mathfrak{g})$-modules M, M', M'' these morphisms satisfy the braid equation

$$(c^f_{M',M''} \otimes \mathrm{id}_M)(\mathrm{id}_{M'} \otimes c^f_{M,M''})(c^f_{M,M'} \otimes \mathrm{id}_{M''}) = \\ = (\mathrm{id}_{M''} \otimes c^f_{M,M'})(c^f_{M,M''} \otimes \mathrm{id}_{M'})(\mathrm{id}_M \otimes c^f_{M',M''}).$$

In particular for every integrable module M the morphism $c^f_{M,M}$ is a braiding on M.

Lemma 1.3.8. The braidings defined in Example 1.3.7 are left and right triangular.

Proof. We will construct a basis B of M such that the braiding Θ is left triangular with respect to this basis. Consider the total order $\rhd$ defined on V using basis $\Pi = \{\mu_1, \dots, \mu_s\}$ of the root system in the following way:

$$\sum_{i=1}^{s} a_i\mu_i \rhd \sum_{i=1}^{s} b_i\mu_i \Leftrightarrow (a_1, \dots, a_s) > (b_1, \dots, b_s),$$

where on the right side we order the sequences in $\mathbb{R}^s$ lexicographically by identifying them with words of s letters from $\mathbb{R}$ (for a definition of the lexicographical order see Section 2.1).
Then for $\mu, \mu' \in V, \nu \in \mathbb{Z}\Phi$, $\mu \rhd \mu'$ implies $\mu + \nu \rhd \mu' + \nu$ and $\nu > 0$ implies $\nu \rhd 0$. For every $\mu \in \Lambda$ with $M_\mu \neq 0$ choose a totally ordered basis $(B_\mu, \leq)$ of M_μ and order the union $B = \cup_\mu B_\mu$ by requiring that for $b \in B_\mu, b' \in B_{\mu'}, \mu \neq \mu'$

$$b < b' \Leftrightarrow \mu \rhd \mu'.$$

This defines a totally ordered basis of M and for $b \in B_\mu, b' \in B_{\mu'}$ we have

$$\Theta(b \otimes b') = f(\mu', \mu) \left(b' \otimes b + \sum_{\nu > 0} \Theta_v(b' \otimes b) \right) \in f(\mu', \mu) b' \otimes b + \sum_{\nu \lhd \mu'} M_\nu \otimes M,$$

showing that the braiding is indeed left triangular. In the same way one sees that the braiding is also right triangular. $\square$

Example 1.3.9. Assume that k is algebraically closed. Let G be an abelian group and $V \in {}^G_G\mathcal{YD}$ a finite-dimensional Yetter-Drinfeld module over G. Then the induced braiding

$$c : V \otimes V \to V \otimes V,\ \ c(v \otimes w) = v_{(-1)}w \otimes v_{(0)}$$

is left triangular.

Proof. For all $g \in G$ let $V_g := \{v \in V | \delta(v) = g \otimes v\}$. Then the V_g are G-submodules of V. Since every simple submodule of a finite-dimensional G-module is one-dimensional we see that each V_g has a flag of invariant subspaces. So for each $g \in G$ we find a basis $v_1^g, \dots, v_{r_g}^g$ of V_g such that for all $h \in G$

$$h \cdot v_i^g \in kv_i^g \oplus \dots \oplus kv_{r_g}^g.$$

Now by concatenating these bases and ordering each according to the indices we obtain a totally ordered basis such that c is triangular. □

Remark 1.3.10. There are braidings that are triangular but not of diagonal type. For example the braiding on the simple two-dimensional $U_q(\mathfrak{sl}_2)$ module (M, c) of type $+1$ is left and right triangular, but not diagonal. Observe that if c were diagonal with respect to some basis A and diagonal coefficients $\alpha_{a,b}, a, b \in A$, then c would be diagonalizable as endomorphism of $M \otimes M$ with eigenvalues $\pm\sqrt{\alpha_{a,b}\alpha_{b,a}}$ for $b \neq a$ (eigenvectors $\sqrt{\alpha_{b,a}}a \otimes b \pm \sqrt{\alpha_{a,b}}b \otimes a$) resp. $\alpha_{a,a}$. But the eigenvalues of c in our case are -1 and q^{-2} (if $f(\frac{\alpha}{2}, \frac{\alpha}{2}) = q^{-2}$). As we assumed that q is not a root of unity, the braiding cannot be diagonal.

1.3.3 The braid group

One motivation of the braid equation is that a braided vector space always induces representations of the braid groups $\mathbb{B}_n, n \geq 2$. These representations can be used to compute invariants of tangles, knots and links. They also allow us to define generalizations of symmetrizer maps that will play an important role in the theory of Nichols algebras.

Definition 1.3.11. Let $n \geq 2$. The *braid group* $\mathbb{B}_n$ is the group generated by the symbols $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, 1 \leq i \leq n-2 \text{ and}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, 1 \leq i, j \leq n-1, |i-j| \geq 2.$$

Remark 1.3.12. If (V, c) is a braided vector space we have a unique representation $\rho : \mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n})$ such that for $1 \leq i \leq n-1$

$$\rho(\sigma_i) = \mathrm{id}_{V^{\otimes i-1}} \otimes c \otimes \mathrm{id}_{V^{\otimes n-i-1}}.$$

There is a natural projection $\pi : \mathbb{B}_n \to \mathbb{S}_n$ into the symmetric group, sending σ_i to the transposition $(i, i+1)$. This projection has a set-theoretical section $s : \mathbb{S}_n \to \mathbb{B}_n$, called the Matsumoto section, such that if $\omega = \tau_1 \dots \tau_r$ is a reduced expression of ω with $\tau_i = (j_i, j_i + 1)$, then

$$s(\omega) = \sigma_{j_1} \dots \sigma_{j_r}.$$

Definition 1.3.13. Let (V, c) be a braided vector space and $n \geq 2$. The map

$$\mathfrak{S}_n := \sum_{\sigma \in \mathbb{S}_n} \rho\left(s(\sigma)\right) \in \operatorname{End}(V^{\otimes n})$$

is called the *n-th quantum symmetrizer map.*

1.4 Braided Hopf algebras

1.4.1 Definition and examples

Braided bialgebras and braided Hopf algebras play an important role in the structure theory of pointed Hopf algebras [5]. Although quite a lot is known about braided Hopf algebras in general [44], there are many open problems, especially in the theory of Nichols algebras [1].
Usually braided bialgebras are defined within the context of a braided category. However sometimes a non-categorical point of view provides additional information [2]. We give two definitions reflecting the two points of view, where in the categorical setting we restrict ourselves to the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode.
We will start in the general setting without referring to a braided category.

Definition 1.4.1. Let (V, c) be a braided vector space and let $f : V^{\otimes n} \to V^{\otimes m}$ be a linear transformation. f *commutes with* c if

$$c_{1,m}(\operatorname{id}_V \otimes f) = (f \otimes \operatorname{id}_V)c_{1,n} \text{ and } c_{m,1}(f \otimes \operatorname{id}_V) = (\operatorname{id}_V \otimes f)c_{n,1}.$$

A *braided algebra* is a quadruple (A, ∇, η, c), where (A, ∇, η) is an algebra and (A, c) is a braided vector space such that ∇, η commute with c.
A *braided coalgebra* is a quadruple $(C, \Delta, \varepsilon, c)$, where (C, Δ, ε) is an algebra and (C, c) is a braided vector space such that Δ, ε commute with c.

Assume that (A, ∇, η, c) is a braided algebra. Then $A\underline{\otimes}A := A \otimes A$ together with the unit $\eta_A \otimes \eta_A$, multiplication

$$\nabla_{A\underline{\otimes}A} := (\nabla_A \otimes \nabla_A)(\operatorname{id}_A \otimes c \otimes \operatorname{id}_A)$$

and braiding $c_{2,2}$ is again a braided algebra. Dually, if $(C, \Delta, \varepsilon, c)$ is a braided coalgebra, then $C\underline{\otimes}C := C \otimes C$ together with the counit $\varepsilon_C \otimes \varepsilon_C$, comultiplication

$$\Delta_{C\underline{\otimes}C} := (\mathrm{id}_C \otimes c \otimes \mathrm{id}_C)(\Delta_C \otimes \Delta_C)$$

and braiding $c_{2,2}$ is again a braided coalgebra.

Definition 1.4.2. A *braided bialgebra* is a tuple $(R, \nabla, \eta, \Delta, \varepsilon, c)$ such that (R, ∇, η, c) is a braided algebra, $(R, \Delta, \varepsilon, c)$ is a braided coalgebra and one of the following equivalent conditions is satisfied:

- $\Delta : R \to R\underline{\otimes}R$, $\varepsilon : R \to k$ are algebra morphisms.
- $\nabla : R\underline{\otimes}R \to R$, $\eta : k \to R$ are coalgebra morphisms.

R is called a *braided Hopf algebra* if the identity map id_R is invertible in the convolution algebra $\mathrm{End}_k(R)$, i.e. there is a map $\mathbf{S} \in \mathrm{End}_k(R)$ (the *antipode* of R) such that for all $r \in R$

$$r_{(1)}\mathbf{S}(r_{(2)}) = \varepsilon(r)1 = \mathbf{S}(r_{(1)})r_{(2)}.$$

If the antipode exists, it commutes with c [44]. A morphism of braided bialgebras is a morphism of algebras, coalgebras and braided vector spaces.

Now we will define Hopf algebras in a Yetter-Drinfeld category.

Definition 1.4.3. Let H be a Hopf algebra with bijective antipode.
An *algebra* (A, ∇, η) *in* ${}^H_H\mathcal{YD}$ is an algebra (A, ∇, η) such that $A \in {}^H_H\mathcal{YD}$ and ∇ and η are morphisms in ${}^H_H\mathcal{YD}$ (i.e. linear and colinear).
A *coalgebra* (C, Δ, ε) *in* ${}^H_H\mathcal{YD}$ is a coalgebra (C, Δ, ε) such that $C \in {}^H_H\mathcal{YD}$ and Δ and ε are morphisms in ${}^H_H\mathcal{YD}$.

Algebras (resp. coalgebras) in ${}^H_H\mathcal{YD}$ are braided algebras (resp. braided coalgebras) with the induced Yetter-Drinfeld braiding. Thus we can form braided tensor product algebras and coalgebras as above. These are the again algebras (resp. coalgebras) in ${}^H_H\mathcal{YD}$.

Definition 1.4.4. Let H be a Hopf algebra with bijective antipode. A *braided bialgebra in* ${}^H_H\mathcal{YD}$ is a quintuple $(R, \nabla, \eta, \Delta, \varepsilon)$ such that (R, ∇, η) is an algebra in ${}^H_H\mathcal{YD}$, (R, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$ and one of the following equivalent conditions is satisfied:

- $\Delta : R \to R\underline{\otimes}R$, $\varepsilon : R \to k$ are algebra morphisms.
- $\nabla : R\underline{\otimes}R \to R$, $\eta : k \to R$ are coalgebra morphisms.

R is called a *Hopf algebra in* ${}^H_H\mathcal{YD}$ if the identity map id_R is invertible in the convolution algebra $\mathrm{End}_k(R)$, i.e. there is a map $\mathbf{S} \in \mathrm{End}_k(R)$ (the *antipode* of R) such that for all $r \in R$

$$r_{(1)}\mathbf{S}(r_{(2)}) = \varepsilon(r)1 = \mathbf{S}(r_{(1)})r_{(2)}.$$

If the antipode exists, it is linear and colinear [44]. A morphism $\phi : R \to R'$ of braided bialgebras in ${}^H_H\mathcal{YD}$ is a morphism of algebras and coalgebras that is also a morphism in the category ${}^H_H\mathcal{YD}$ (i.e. H-linear and H-colinear).

For a braided Hopf algebra $(R, \nabla_R, \eta_R, \Delta_R, \varepsilon_R, c)$ the antipode is an anti-algebra morphism in the sense that it is an algebra morphism from R into $R^{op,c}$, where $R^{op,c}$ has unit map η_R and multiplication $\nabla_R\, c$.

Remark 1.4.5. Let H be a Hopf algebra with bijective antipode. Every bialgebra in ${}^H_H\mathcal{YD}$ is a braided bialgebra in the sense of the first definition. Conversely Takeuchi shows that every rigid braided bialgebra can be realized as a bialgebra in the category of Yetter-Drinfeld modules over some Hopf algebra H with bijective antipode [44].
Nevertheless our notion of a morphism of braided bialgebras is weaker than that of a morphism of bialgebras in a Yetter-Drinfeld category. Assume that we have a bialgebra R in the category of Yetter-Drinfeld modules over some Hopf algebra. A subbialgebra R' in this setting is a Yetter-Drinfeld submodule and thus we have induced braidings

$$R' \otimes R \to R \otimes R', R' \otimes R \to R \otimes R' \text{ and } R' \otimes R' \to R' \otimes R'.$$

On the other hand assume we have a braided bialgebra R'' that is a braided subbialgebra of R in the sense that the inclusion is a morphism of braided bialgebras, but R'' is not necessarily a Yetter-Drinfeld submodule. In this case we obtain only a braiding for R''

$$R'' \otimes R'' \to R'' \otimes R''.$$

Takeuchi calls R'' a non-categorical (braided) subbialgebra of R in this case.

Example 1.4.6. Let (V, c) be a braided vector space. Then the tensor algebra

$$T_c(V) := k1 \oplus \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$$

is a braided vector space with braiding given by the homogeneous components

$$c_{n,m} : V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes m} \otimes V^{\otimes n}.$$

It becomes a braided Hopf algebra with comultiplication, counit and antipode given for all $v \in V$ by

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \;\; \varepsilon(v) = 0, \;\; \mathbf{S}(v) = -v.$$

If M is a Yetter-Drinfeld module over H with Yetter-Drinfeld braiding c, then $T_c(M)$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with the usual tensor product structures.

The braided tensor algebra from this example is isomorphic (as algebra) to the usual tensor algebra. The next step is to define a generalization of the symmetric algebra, the so-called Nichols algebra of a braided vector space. The structure of these algebras is often much more complicated than that of the usual symmetric algebra and it is a central task of this thesis to determine the structure for a class of braidings.

Definition 1.4.7. Let (V, c) be a braided vector space. The *Nichols algebra* $\mathcal{B}(V, c)$ is a braided Hopf algebra (its braiding is denoted by $c_{\mathcal{B}(V,c)}$, or just by c later) with the following properties:

- $\mathcal{B}(V, c) = \bigoplus_{n \geq 0} \mathcal{B}(V, c)(n)$ is graded as algebra and coalgebra,
- $c_{\mathcal{B}(V,c)} \left(\mathcal{B}(V, c)(m) \otimes \mathcal{B}(V, c)(n)\right) \subset \mathcal{B}(V, c)(n) \otimes \mathcal{B}(V, c)(m)$,
- $\mathcal{B}(V, c)$ is generated by $\mathcal{B}(V, c)(1)$,
- $\mathcal{B}(V, c)(1) = P(\mathcal{B}(V, c))$ and
- $V \simeq P(\mathcal{B}(V, c))$ as braided vector spaces.

Proposition 1.4.8. For every braided vector space (V, c) there is a Nichols algebra and it is unique up to isomorphism. The Nichols algebra can be constructed in the following way: Let I be the sum of all ideals of $T_c(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are also coideals. Then $\mathcal{B}(V, c) := T_c(V)/I$ is the factor algebra and factor coalgebra. Actually we know a bit more about the ideal I, namely it is given by the kernels of the quantum symmetrizers from Definition 1.3.13

$$I = \bigoplus_{n \geq 2} \ker \mathfrak{S}_n.$$

In particular, if M is a Yetter-Drinfeld module over H with Yetter-Drinfeld braiding c, then $\mathcal{B}(M, c)$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$.

Proof. See the survey article [5]. □

So in order to know the Nichols algebra it would in principle be sufficient to know the kernels of quantum symmetrizer maps. An interesting approach was taken by Flores de Chela and Green in [10], where they compute the determinant of certain restrictions of the quantum symmetrizers for braidings of diagonal type. In general however it is not possible to determine all these kernels explicitly.

Example 1.4.9. Assume that (V, c) is braided vector space of Hecke type, i.e. there is a scalar $q \in k$ such that

$$(c + \mathrm{id})(c - q\,\mathrm{id}) = 0.$$

Then the ideal I is generated by $\ker \mathfrak{S}_2 = \mathrm{Im}(c - q\,\mathrm{id})$. For a proof also see [5].

Example 1.4.10. Let I be a finite set, $(a_{ij})_{i,j\in I}$ a generalized Cartan matrix and assume that there are relatively prime positive integers $(d_i)_{i\in I}$ such that for all $i, j \in I$

$$d_i a_{ij} = d_j a_{ji}.$$

Let $q \in k$ be not a root of unity and $V := \oplus_{i\in I} ki$. Define a braiding c of diagonal type on V by

$$c(i \otimes j) := q^{d_i a_{ij}} j \otimes i \ \ \forall i, j \in I.$$

The Nichols algebra $\mathcal{B}(V, c)$ is the algebra $\mathbf{f}$ from Lusztigs book [27], which is isomorphic to the positive part $U_q^+(\mathfrak{g})$, if (a_{ij}) is the Cartan matrix of $\mathfrak{g}$.

Proof. See [4, Proposition 2.7]. □

In the example above we see that Nichols algebras occur in the theory of quantum groups. In Chapter 4 we will see that not only Nichols algebras of braidings of diagonal type occur as subalgebras of quantum groups, but also Nichols algebras of certain $U_q(\mathfrak{g})$-modules.

1.4.2 Radford biproducts and Hopf algebras with a projection

In the theory of pointed Hopf algebras braided Hopf algebras and Nichols algebras occur in the context of Radford biproducts [34]. In order to distinguish comultiplications in usual Hopf algebras from those in braided Hopf algebras, we use Sweedler notation with upper indices for braided Hopf algebras

$$\Delta_R(r) = r^{(1)} \otimes r^{(2)}.$$

Definition 1.4.11. Let H be a Hopf algebra with bijective antipode and R a braided Hopf algebra in ${}^H_H\mathcal{YD}$. Then we can turn $R\#H := R \otimes H$ into a bialgebra by using the crossed product

$$(r\#h)(r'\#h') = r(h_{(1)} \cdot r')\#h_{(2)}h'$$

with unit $1_R\#1_H$ and the crossed coproduct

$$\Delta(r\#h) = r^{(1)}\#r^{(2)}{}_{(-1)}h_{(1)} \otimes r^{(2)}{}_{(0)}\#h_{(2)}$$

with counit $\varepsilon_R \otimes \varepsilon_H$. This bialgebra is actually a Hopf algebra with antipode

$$\mathbf{S}_{R\#H}(r\#h) = \left(1\#\mathbf{S}_H\left(r_{(-1)}h\right)\right)\left(\mathbf{S}_R\left(r_{(0)}\right)\#1\right),$$

and it is called the *Radford biproduct* of R and H.

In the situation of the definition we have a Hopf algebra projection

$$\pi : R\#H \to H,\ \ \pi(r\#h) = \varepsilon(r)h.$$

R is a subalgebra of $R\#H$ and H is a Hopf subalgebra of $R\#H$.
As we have now seen, Radford biproducts are Hopf algebras that have a projection onto a Hopf subalgebra. A theorem by Radford says that also the converse is true. Let A, H be Hopf algebras and assume there is a Hopf algebra injection $\iota : H \to A$ and a Hopf algebra projection $\pi : A \to H$ such that $\pi\iota = \mathrm{id}_H$. In this case the algebra of right coinvariants with respect to π,

$$R := A^{\mathrm{co}\,\pi} := \{a \in A | (\mathrm{id}_A \otimes \pi)\Delta(a) = a \otimes 1\},$$

is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, where the action is the restriction of the adjoint action and the coaction and comultiplication are given by

$$\delta_R(r) = \pi(r_{(1)}) \otimes r_{(2)} \ \text{ and}$$
$$\Delta_R(r) = r_{(1)}\iota\mathbf{S}_H\pi(r_{(2)}) \otimes r_{(3)} \ \text{ for all } r \in R.$$

The antipode is

$$\mathbf{S}_R(r) = \pi\left(r_{(1)}\right)\mathbf{S}_A\left(r_{(2)}\right).$$

Define a linear map by

$$\theta : A \to R,\ \ \theta(a) = a_{(1)}\iota\mathbf{S}_H\pi(a_{(2)}).$$

θ is a coalgebra projection onto R and we have the following theorem.

Theorem 1.4.12. [34, 5] The maps

$$\begin{aligned} A \to R\#H, &\quad a \mapsto \theta(a_{(1)})\#\pi(a_{(2)}) \text{ and} \\ R\#H \to A, &\quad r\#h \mapsto r\iota(h) \end{aligned}$$

are mutually inverse isomorphisms of Hopf algebras.

1.4.3 Braided Hopf algebras of triangular type

In this section braided bialgebras of triangular type are introduced. These are the objects we will mainly deal with.

Remark 1.4.13. Let R be a braided bialgebra with braiding c and $P(R)$ the space of primitive elements. Then $P(R)$ is a *braided subspace* of R, i.e.

$$c(P(R) \otimes P(R)) = P(R) \otimes P(R).$$

Proof. This follows from the fact that c and c^{-1} commute with Δ and η. $\square$

Definition 1.4.14. A braided bialgebra $(R, \nabla, \eta, \Delta, \varepsilon, c)$ will be called *of left resp. right triangular type* if it is generated as an algebra by a finite-dimensional braided subspace $V \subset P(R)$ and the braiding on V is left resp. right triangular.

The central examples are Nichols algebras of integrable $U_q(\mathfrak{g})$-modules and of Yetter-Drinfeld modules over abelian groups. We will now show that every braided bialgebra generated by primitive elements is a quotient of the braided tensor bialgebra and describe these quotients in terms of braided biideals.

Lemma 1.4.15. Let (R, c_R) be a braided bialgebra, $V \subset P(R)$ a braided subspace. Then there is a unique homomorphism of braided bialgebras $\pi : T(V, c_R|V \otimes V) \to R$ with $\pi|V = \mathrm{id}_V$.

Proof. Uniqueness is obvious. Of course π exists as algebra homomorphism. Denote the braiding on the tensor algebra induced by $c_R|V \otimes V$ by $c_{T(V)}$. Using the universal property of the tensor algebra we obtain that π is a coalgebra homomorphism, provided $\pi \otimes \pi : T(V)\underline{\otimes}T(V) \to R\underline{\otimes}R$ is an algebra homomorphism. It is easy to check this, if $(\pi \otimes \pi)c_{T(V)} = c_R(\pi \otimes \pi)$. So we are left to show this. By construction we have $\pi|V^{\otimes l} = m_l|V^{\otimes l}$, a restriction of the l-fold multiplication of R. Thus for all $r, s \geq 0$

$$\begin{aligned}
(\pi \otimes \pi)c_{T(V)}|V^{\otimes r} \otimes V^{\otimes s} &= (m_s \otimes m_r)(c_{T(V)})_{r,s}|V^{\otimes r} \otimes V^{\otimes s} \\
&= c_R(m_r \otimes m_s)|V^{\otimes r} \otimes V^{\otimes s} \\
&= c_R(\pi \otimes \pi)|V^{\otimes r} \otimes V^{\otimes s},
\end{aligned}$$

where the second equality is because the multiplication of R commutes with c. $\square$

Definition 1.4.16. Let (R, c) be a braided bialgebra. A subspace $I \subset R$ is called a *braided biideal*, if it is an ideal, a coideal and

$$c(R \otimes I + I \otimes R) = R \otimes I + I \otimes R.$$

If (R, c) is a braided Hopf algebra with antipode $\mathbf{S}$, I is called a braided Hopf ideal if it is a braided biideal with $\mathbf{S}(I) \subset I$.

Lemma 1.4.17. Let R be a braided bialgebra with braiding c.

1. If $I \subset R$ is a braided biideal there is a unique structure of a braided bialgebra on the quotient R/I such that the canonical map is a homomorphism of braided bialgebras.

2. If $\pi : (R, c) \to (S, d)$ is a morphism of braided bialgebras, $\ker \pi$ is a braided biideal of R.

3. Analogous statements hold for braided Hopf ideals.

Proof. Part 1: Uniqueness is clear because π is surjective. Obviously R/I is an algebra and a coalgebra in the usual way with structure maps $\bar{m}, \bar{\eta}, \bar{\Delta}$ and $\bar{\varepsilon}$. Furthermore $c(\ker(\pi \otimes \pi)) = \ker(\pi \otimes \pi)$ and thus c induces an automorphism $\bar{c}$ of $R/I \otimes R/I$ such that $(\pi \otimes \pi)c = \bar{c}(\pi \otimes \pi)$. Surjectivity of π ensures that $\bar{c}$ satisfies the braid equation and that $\bar{\Delta}, \bar{\varepsilon}$ are algebra homomorphisms. $\bar{m}, \bar{\eta}, \bar{\Delta}, \bar{\varepsilon}$ commute with $\bar{c}$ because $m, \eta, \Delta, \varepsilon$ commute with c and π is surjective.
Part 2: Of course $I := \ker \pi$ is an ideal and a coideal. It remains to show that the condition for c holds. As $(\pi \otimes \pi)c = d(\pi \otimes \pi)$ and c is bijective we have $c(\ker(\pi \otimes \pi)) = \ker(\pi \otimes \pi)$. In view of $\ker(\pi \otimes \pi) = I \otimes R + R \otimes I$ the proof is complete. □

Chapter 2

Lyndon words and PBW bases for braided Hopf algebras of triangular type

Starting from the theory of free Lie algebras, in particular Shirshov's basis for free Lie algebras [42], Lalonde and Ram [24] proved in 1995 that every Lie algebra $\mathfrak{g}$ generated by an ordered set X has a basis that can be described by certain Lyndon words in the letters X. Their result gives a description of the PBW basis of $U(\mathfrak{g})$ in terms of the generators of $\mathfrak{g}$. Kharchenko [20] showed that an analogous PBW result can be obtained for a class of pointed Hopf algebras which he calls character Hopf algebras. In fact Kharchenko's result can be viewed in the setting of braided Hopf algebras with diagonal braidings that are generated by primitive elements.
The main Theorem 2.2.4 of this chapter shows that Kharchenko's result is actually true for a much larger class of braided Hopf algebras, namely braided Hopf algebras of triangular type.
The setting of triangular braidings is the natural context for our proof of the PBW theorem, which basically follows Kharchenko's approach. Nevertheless the situation is more complicated than in the diagonal case and new methods are needed. We do not obtain the whole strength of Kharchenko's results in some details (see Remark 2.2.5).
Apart from the fact that the PBW theorem offers an interesting insight into the structure of braided Hopf algebras of triangular type it allows us to treat examples explicitly. We apply the theorem to determine the structure of Nichols algebras of low-dimensional $U_q(\mathfrak{sl}_2)$-modules. Moreover in view of Example 1.3.9 we can generalize Kharchenko's original result to arbitrary Hopf algebras that are generated by an abelian group and a finite set of skew primitive elements.

2.1 Lyndon words and braided commutators

The PBW basis will be described in terms of Lyndon words in the generators. Here we will present the definition and basic facts about these words. Let $(X,<)$ be a finite totally ordered set and $\mathbb{X}$ the set of all words in the letters X (the free monoid over X). Recall that the *lexicographical order* on $\mathbb{X}$ is the total order defined in the following way: For words $u, v \in \mathbb{X}$, $u<v$ iff either $v \in u\mathbb{X}$ (u is the beginning of v) or if there exist $r, s, t \in \mathbb{X}, a, b \in X$ such that

$$u = ras, v = rbt \text{ and } a < b.$$

For example if $x, y \in X, x<y$ then $x<xy<y$.

Notation 2.1.1. For a word $u \in \mathbb{X}$ let $l(u)$ be the length of u. Define for $n \in \mathbb{N}, v \in \mathbb{X}$ the following subsets of $\mathbb{X}$:

$$\begin{aligned}
\mathbb{X}^n &:= \{u \in \mathbb{X} | l(u) = n\}, \\
\mathbb{X}_{>v} &:= \{u \in \mathbb{X} | u>v\}, \\
\mathbb{X}_{\geq v} &:= \{u \in \mathbb{X} | u \geq v\}, \\
\mathbb{X}^n_{>v} &:= \mathbb{X}^n \cap \mathbb{X}_{>v}, \text{ and} \\
\mathbb{X}^n_{\geq v} &:= \mathbb{X}^n \cap \mathbb{X}_{\geq v}.
\end{aligned}$$

Definition 2.1.2. Let $u \in \mathbb{X}$. The word u is called a *Lyndon word* if $u \neq 1$ and u is smaller than any of its proper endings. This means for all $v, w \in \mathbb{X} \setminus \{1\}$ such that $u = vw$ we have $u<w$.
These words are also called regular words in [47] or standard words in [42, 20].

A word u is Lyndon if and only if for every factorization $u = vw$ of u into non-empty words v, w we have $u = vw<wv$ [26, 5.1.2.].

Example 2.1.3. Let $a \in X$. Then a is Lyndon, but for $n \geq 2$ the word a^n is not Lyndon.
If $a, b \in X, a < b$ all words of the form $a^n b^m$ with $n \geq 2, m \geq 1$ are Lyndon. Concrete examples of more complicated Lyndon words are

$$a^2bab, a^2babab, a^2bababab, \ldots$$

The following two theorems will provide important factorization properties of Lyndon words.

Theorem 2.1.4. (Lyndon, [26, Theorem 5.1.5.])
Any word $u \in \mathbb{X}$ may be written uniquely as a non-increasing product of Lyndon words

$$u = l_1 l_2 \ldots l_r, \;\; l_i \text{ Lyndon words and } l_1 \geq l_2 \geq \ldots \geq l_r.$$

This decomposition is obtained inductively by choosing l_1 to be the longest beginning of u that is a Lyndon word. It will be referred to as the *Lyndon decomposition* of u. The occurring Lyndon words are called the *Lyndon letters* of u.

Example 2.1.5. Let $a, b, c \in X, a < b < c$. The Lyndon decomposition of the word $c^2abaca^2cba^2bab$ is

$$(c)(c)(abac)(a^2cb)(a^2bab).$$

Theorem 2.1.6. [35, Theorem 5.1. and section 4.1.]
The set of Lyndon words is a Hall set with respect to the lexicographical order. This means that for every Lyndon word $u \in \mathbb{X} \setminus X$ we have a fixed decomposition $u = u'u''$ into non-empty Lyndon words u', u'' such that either $u' \in X$ or the decomposition of u' has the form $u' = vw$ with $w \geq u''$.

This decomposition is obtained by choosing u'' to be the minimal (with respect to the lexicographical order) or (equivalently) the longest proper end of u that is Lyndon. As in [20] it is referred to as the *Shirshov decomposition* of u.

Example 2.1.7. For a Lyndon word u define its associated non-associative word (u) - an element of the free magma as defined by Serre [41, Part I, Chapter 4.1] - inductively on the length of u. For $x \in X$ set $(x) := x$. For a word $u \in \mathbb{X} \setminus X$ with Shirshov decomposition $u = u'u''$ let $(u) := ((u'), (u''))$ be the ordered pair of the non-associative words associated to u' and u''. For $a, b \in X$ with $a < b$ we have then

$$(ab) = (a, b),$$

$$(a^2b) = (a, (ab)) = (a, (a, b)),$$
$$(abac) = ((ab), (ac)) = ((a, b), (a, c)), \text{ and}$$
$$(a^2babab) = (((a, (a, b)), (a, b)), (a, b)).$$

For example, a^2babab has Shirshov decomposition $(a^2bab)(ab)$.

Major tools for constructing the PBW basis will be iterated braided commutators. These are defined in a similar way as the non-associative words above by replacing the brackets of a non-associative word by a skew-commutator that involves the braiding. Take a finite-dimensional vector-space V, an endomorphism r of $V \otimes V$ satisfying the braid equation and assume that X is a basis of V. Define the endomorphism $r_{n,m} : V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes m} \otimes V^{\otimes n}$ in

the same way as for braidings. We will omit the indices n, m whenever it is clear from the context which endomorphism is used.
In the following definition we identify $k\mathbb{X}$ - the free algebra over X - with the tensor algebra of V in the obvious way. We construct a k-linear endomorphism $[-]_r$ of $k\mathbb{X}$ inductively.

Definition 2.1.8. Set for all $x \in X$

$$[1]_r := 1 \text{ and } [x]_r := x.$$

For Lyndon words $u \in \mathbb{X}$ of degree > 1 with Shirshov decomposition $u = vw$ define

$$[u]_r := m(\mathrm{id} - r_{l(v),l(w)})([v]_r \otimes [w]_r),$$

where m denotes multiplication in $k\mathbb{X}$. For an arbitrary word with Lyndon decomposition $u = u_1 \dots u_t$ let

$$[u]_r := [u_1]_r \dots [u_t]_r.$$

Obviously $[-]_r$ is a graded homomorphism of the graded vector space $k\mathbb{X}$. The idea of using a homomorphism of this type to construct PBW bases can be found in [20] and is motivated by the theory of Lie algebras [24]. Furthermore Ringel [36] and Leclerc [25] constructed PBW bases for deformed enveloping algebras made up of iterated commutators. These iterated commutators are also closely connected to Lyndon words.
Finally we give a lemma from [20] that provides a good tool for comparing words using their Lyndon decompositions.

Lemma 2.1.9. [20, Lemma 5]
For $u, v \in \mathbb{X}$ we have $u<v$ if and only if u is smaller than v when comparing them using the lexicographical order on the Lyndon letters. This means if $v = l_1 \dots l_r$ is the Lyndon decomposition of v, we have $u<v$ iff

- u has Lyndon decomposition $u = l_1 \dots l_i$ for some $0 \leq i < r$
- or u has Lyndon decomposition $u = l_1 \dots l_{i-1} \cdot l \cdot l'_{i+1} \dots l'_s$ for some $s \in \mathbb{N}, 1 \leq i \leq r$ and some Lyndon words $l, l'_{i+1}, \dots, l'_s$ with $l<l_i$.

2.2 The PBW theorem

Now we can formulate the PBW theorem for braided Hopf algebras of triangular type. First we will give a formal definition of the term PBW basis.

Definition 2.2.1. Let A be an algebra, $P, S \subset A$ subsets and $h : S \to \mathbb{N} \cup \{\infty\}$ a map. Assume that $<$ is total ordering of S and let $B(P, S, <, h)$ be the set of all products

$$s_1^{e_1} \dots s_t^{e_t} p$$

with $t \in \mathbb{N}_0$,$s_1 > \dots > s_t$, $s_i \in S$, $0 < e_i < h(s_i)$ and $p \in P$. This set is called the *PBW set* generated by P, $(S, <)$ and h. h is called the height function of the PBW set.
We say $(P, S, <, h)$ is *PBW basis* of A if $B(P, S, <, h)$ is a basis of A.

Of course every algebra A has the trivial PBW basis with $S = \emptyset$ and P a basis of A. If H is a pointed (braided) bialgebra we are interested in the case $P = G(H)$. Thus in this chapter we are interested in the case where $P = \{1\}$. We will say that $B(S, <, h) := B(\{1\}, S, <, h)$ is the PBW set (resp. PBW basis) generated by $(S, <)$ and h.
Fix a finite-dimensional braided vector space (V, c) which is left triangular with respect to a basis X of V. Let d be the diagonal component of c and abbreviate $[-] := [-]_{d^{-1}}$. Identify $T(V)$ with $k\mathbb{X}$.

Definition 2.2.2. Define the *standard order* on $\mathbb{X}$ in the following way. For two elements $u, v \in \mathbb{X}$ write $u \succ v$ if and only if u is shorter than v or if $l(u) = l(v)$ and $u > v$ lexicographically.

In this order the empty word 1 is the maximal element. As X is assumed to be finite, this order fulfills the ascending chain condition, making way for inductive proofs. Define $\mathbb{X}_{\succ u}, \mathbb{X}_{\succeq u}$ etc. in the obvious way.
Now we will define the PBW set that will lead to the PBW basis of our braided Hopf algebra. The sets S_I resp. B_I are analogues of the sets of "hard superletters" resp. of "restricted monotonous words in hard superletters" found in [20].

Definition 2.2.3. Let $I \subsetneq k\mathbb{X}$ be a biideal. Let S_I be the set of Lyndon words from $\mathbb{X}$ that do not appear as (standard-) smallest monomial in elements of I:

$$S_I := \{u \in \mathbb{X} | \text{ u is a Lyndon word and } u \notin k\mathbb{X}_{\succ u} + I\}.$$

For $u \in S_I$ define the height $h_I(u) \in \{2, 3, \dots, \infty\}$ by

$$h_I(u) := \min\{t \in \mathbb{N} | u^t \in k\mathbb{X}_{\succ u^t} + I\}$$

and let $B_I := B(S_I, <, h_I)$ be the PBW set generated by $(S_I, <)$ and h_I, where $<$ denotes the lexicographical order.

If r is an endomorphism of $V \otimes V$ satisfying the braid equation and $U \subset \mathbb{X}$ is any subset define $[U]_r := \{[u]_r | u \in U\}$. Denote by $k[U]_r$ the k-linear subspace of $k\mathbb{X}$ spanned by $[U]_r$ (To avoid confusion with the notation for polynomial rings let me note that no polynomial rings will be considered during this section).
One of the central results of this thesis is the following theorem. Note that in the special case of diagonal braidings this theorem together with Lemma 2.2.6 is a braided analogue of [20, Theorem 2].

Theorem 2.2.4. Let (V, c) be a finite-dimensional braided vector space that is left triangular with respect to some basis X. Identify $T(V)$ with $k\mathbb{X}$ and let $I \subsetneq k\mathbb{X}$ be a braided biideal, $\pi : k\mathbb{X} \to (k\mathbb{X})/I$ the quotient map. Then $\pi(B_I)$ and $\pi([B_I]_c)$ are bases of $(k\mathbb{X})/I$.
These are the truncated PBW bases generated by $\pi(S_I)$ resp. $\pi([S_I]_c)$ with heights $h_I(u)$ for $u \in S_I$.

The proof will be done in Section 2.3.

Remark 2.2.5. The reader should observe that in changing from diagonal to triangular braidings we lost some information on the basis. Kharchenko shows that in the diagonal case every Lyndon word u with $u \notin S_I$ is (modulo I) a linear combination of

- words of the same degree as u that are non-ascending products in PBW generators lexicographically smaller than u and
- words of degree smaller than that of u that are non-ascending products in arbitrary PBW generators.

It is an open question whether this (or something similar) can be done for triangular braidings.

There is an important result on the possible values of the height function in [20]. A generalization to the situation of triangular braidings is given in the following theorem which will also be proved in Section 2.3.

Theorem 2.2.6. Let (V, c) be a finite-dimensional braided vector space that is left triangular with respect to some basis X. Identify $T(V)$ with $k\mathbb{X}$ and let $I \subsetneq k\mathbb{X}$ be a braided biideal and $v \in S_I$. Define the scalar $\gamma_{v,v} \in k$ by

$$d(v \otimes v) = \gamma_{v,v} v \otimes v,$$

where d is the diagonal component of c and assume $h := h_I(v) < \infty$.

Then $\gamma_{v,v}$ is a root of unity. Let t be the order of $\gamma_{v,v}$. If $\operatorname{char} k = 0$ then $\gamma_{v,v} \neq 1$ and $h = t$. If $\operatorname{char} k = p > 0$ then $h = tp^l$ for some $l \in \mathbb{N}$.

2.3 Proof of the PBW theorem

2.3.1 Combinatorial properties of braided commutators

The proof of the PBW theorem from the preceding section will use combinatorial properties of braided commutators and of the comultiplication. Some of these properties were studied in [20] in the case of diagonal braidings. The central problem of this subsection and the next is to generalize results from the diagonal case and to provide new tools necessary in the triangular case. The next lemma for example is trivial in the diagonal case.

Lemma 2.3.1. Let V be a vector space and assume that $c \in \mathrm{End}(V \otimes V)$ is a left triangular endomorphism with respect to the basis X which satisfies the braid equation. We have for words $u, v \in \mathbb{X}$:

$$c(u \otimes v) \in d(u \otimes v) + k\mathbb{X}^{l(v)}_{>v} \otimes k\mathbb{X}^{l(u)},$$

where d is the diagonal component of c.

Proof. Let $\gamma_{x,y}$ be the diagonal coefficients of c from the definiton of triangular braidings (1.3.5). We use double induction on $l(u)$ and $l(v)$. For $l(u) = 0$ and for $l(v) = 0$ the claim is trivial. Assume $l(u), l(v) > 0$. If $l(u) = l(v) = 1$ the claim is exactly the condition from Definition 1.3.5. Now let $l(u) = 1, l(v) > 1$ and write $v = xw$ with $x \in X, w \in \mathbb{X}$. Use the notation from Definition 1.3.5. Then with $q := l(v)$ the induction hypothesis gives

$$\begin{aligned} c_{1,q}(u \otimes v) &= (\mathrm{id}_V \otimes c_{1,q-1})(c_{1,1}(u \otimes x) \otimes w) \\ &= \gamma_{u,x}(\mathrm{id}_V \otimes c_{1,q-1})((x \otimes u) \otimes w) \\ &\quad + \sum_{z>x} (\mathrm{id}_V \otimes c_{1,q-1})((z \otimes v_{u,x,z}) \otimes w) \\ &\in \gamma_{u,x}\gamma_{u,w} x \otimes w \otimes u + \sum_{z>x} z \otimes k\mathbb{X}^{q-1} \otimes k\mathbb{X}^1 \\ &\subset \gamma_{u,xw} xw \otimes u + k\mathbb{X}^q_{>xw} \otimes k\mathbb{X}^1, \end{aligned}$$

where the last inclusion follows from the definition of the lexicographical order (note that in any case only words of the same length are compared). So now assume $q = l(v) \geq 1, p := l(u) > 1$ and write $u = wx$ for some $x \in X$. Then

$$\begin{aligned} c_{p,q}(u \otimes v) &= (c_{p-1,q} \otimes \mathrm{id}_V)(w \otimes c_{1,q}(x \otimes v)) \\ &\in \gamma_{x,v} c_{p-1,q}(w \otimes v) \otimes x + c_{p-1,q}(w \otimes k\mathbb{X}^q_{>v}) \otimes k\mathbb{X}^1 \\ &\subset \gamma_{x,v}\gamma_{w,v} v \otimes wx + k\mathbb{X}^q_{>v} \otimes k\mathbb{X}^{p-1} \otimes x + k\mathbb{X}^q_{>v} \otimes k\mathbb{X}^p \end{aligned}$$

using the induction hypothesis for p twice. □

Notation 2.3.2. Let (V, c) be a braided vector space that is left triangular with respect to a basis X. An endomorphism r of $V \otimes V$ will be called *admissible* if it satisfies the braid equation and is left triangular with respect to the basis X.

For example the braiding c itself, braidings which are diagonal with respect to the basis X and the zero morphism are admissible. The concept of commutators induced by admissible endomorphism allows us to formulate the process Kharchenko [21] refers to as monomial crystallization, namely the transfer from a basis of iterated commutators to a basis made up of the underlying words. The first part of the following lemma is a generalization of the second part of [20, Lemma 5] to our case of commutators coming from arbitrary admissible endomorphisms.

Lemma 2.3.3. Let (V, c) be a left triangular braided vector space with basis X and r an admissible endomorphism. Then for every word $u \in \mathbb{X}$ the polynomial $[u]_r$ is homogeneous of degree $l(u)$ and the smallest monomial in this term is u with coefficient 1:

$$[u]_r \in u + k\mathbb{X}^{l(u)}_{>u}.$$

In particular if the diagonal component of the braiding c has the coefficients $\gamma_{x,y}$ and r is itself diagonal, we have

$$c([u]_r \otimes [v]_r) \in \gamma_{u,v}[v]_r \otimes [u]_r + k\mathbb{X}^{l(v)}_{>v} \otimes k\mathbb{X}^{l(u)}.$$

Proof. Proceed by induction on $l(u)$. The cases $l(u) = 0, 1$ follow from the definition of $[-]_r$. In the case $l(u) > 1$ first assume u is a Lyndon word. Then we have a Shirshov decomposition $u = vw$ of u. With $p := l(v), q := l(w)$ (m is the multiplication map) we have

$$[u]_r = [v]_r[w]_r - m \circ r_{p,q}([v]_r \otimes [w]_r)$$

and using the induction assumption we obtain

$$\begin{aligned} [u]_r &\in (v + k\mathbb{X}^p_{>v})(w + k\mathbb{X}^q_{>w}) - m \circ r_{p,q}(k\mathbb{X}^p \otimes k\mathbb{X}^q_{\geq w}) \\ &\subset vw + vk\mathbb{X}^q_{>w} + k\mathbb{X}^p_{>v}k\mathbb{X}^q + k\mathbb{X}^q_{\geq w}k\mathbb{X}^p. \end{aligned}$$

From the definition of the lexicographical order we see that the first and second subspace are contained $k\mathbb{X}^{l(u)}_{>u}$. For the third subspace take $a \in \mathbb{X}^q_{\geq w}, b \in$

$\mathbb{X}^p$. Then (because u is Lyndon) $a \geq w > u$ and because a is shorter than u we obtain $ab > u$. Thus

$$[u]_r \in u + k\mathbb{X}^{p+q}_{>u}.$$

Now assume u is not Lyndon. Let $u = u_1 \dots u_t$ be the Lyndon decomposition and let $v := u_1, w := u_2 \dots u_t$ and $p := l(v), q := l(w)$. Then

$$\begin{aligned} [u]_r &= [v]_r[w]_r \in (v + k\mathbb{X}^p_{>v})(w + k\mathbb{X}^q_{>w}) \\ &\subset vw + vk\mathbb{X}^q_{>w} + k\mathbb{X}^p_{>v}k\mathbb{X}^q \subset u + k\mathbb{X}^{p+q}_{>u}. \end{aligned}$$

For the second part observe that if r is diagonal then $[u]_r$ is just a linear combination of words $u' > u$ that are obtained from the word u by permuting the letters of u. If also v' is obtained form v by permuting we have $\gamma_{u',v'} = \gamma_{u,v}$. Thus the diagonal part of c acts on every monomial in $[u]_r \otimes [v]_r$ by multiplication with $\gamma_{u,v}$. Together with the preceding lemma and the first part this completes the proof. □

2.3.2 Combinatorial properties of the comultiplication

Now we will prove some combinatorial properties of the comultiplication in the tensor algebra of a left triangular braided vector space. For this section fix a finite-dimensional braided vector space (V, c), assume that it is left triangular with respect to the basis X and denote by d the diagonal component of c. Abbreviate $[-] := [-]_{d^{-1}}$ using the inverse of the diagonal component (As c is bijective it is easy to prove that the diagonal coefficients of c are not zero).
The technical lemmas in this section are motivated by similar calculations done in [20] in the case of diagonal braidings. A key idea in the step to triangular braidings is to use the iterated commutators induced by the inverse of the diagonal component of the braiding. Just using c or its inverse is not sufficient to prove the main theorem. The following lemma - a generalization of [20, Lemma 8] - requires this. As Kharchenko works with diagonal braidings he can use the inverse of the braiding itself for the commutator. It is a central observation that it does not matter in fact which admissible endomorphism one uses for the commutator. If the final theorem is proved for d^{-1}, it generalizes easily to any other admissible endomorphism (see Remark 2.3.15).

Lemma 2.3.4. Let $u \in \mathbb{X}$ be a Lyndon word and $n := l(u)$. Then

$$\Delta([u]) \in [u]\underline{\otimes}1 + 1\underline{\otimes}[u] + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{>u}\underline{\otimes}k\mathbb{X}^j.$$

Proof. Induction on $n = l(u)$. For $n = 1$ nothing has to be proved. Assume $n > 1$ and let $u = vw$ be the Shirshov decomposition of u. By induction we have

$$\Delta([v]) \in [v]\underline{\otimes}1 + 1\underline{\otimes}[v] + \sum_{\substack{i+j=l(v)\\ i,j\neq 0}} k\mathbb{X}^i_{>v}\underline{\otimes}k\mathbb{X}^j \quad \text{and}$$

$$\Delta([w]) \in [w]\underline{\otimes}1 + 1\underline{\otimes}[w] + \sum_{\substack{l+m=l(w)\\ l,m\neq 0}} k\mathbb{X}^l_{>w}\underline{\otimes}k\mathbb{X}^m.$$

Now we obtain

$$\begin{aligned}
\Delta([v])\Delta([w]) &\in \left([v]\underline{\otimes}1 + 1\underline{\otimes}[v] + \sum_{\substack{i+j=l(v)\\ i,j\neq 0}} k\mathbb{X}^i_{>v}\underline{\otimes}k\mathbb{X}^j\right)\cdot \\
&\quad \left([w]\underline{\otimes}1 + 1\underline{\otimes}[w] + \sum_{\substack{l+m=l(w)\\ l,m\neq 0}} k\mathbb{X}^l_{>w}\underline{\otimes}k\mathbb{X}^m\right) \\
&\subset [v][w]\underline{\otimes}1 + [v]\underline{\otimes}[w] + 1\underline{\otimes}[v][w] + \sum_{\substack{i+j=n\\ i,j\neq 0}} k\mathbb{X}^i_{>u}\underline{\otimes}k\mathbb{X}^j
\end{aligned}$$

using the following facts:
$([v]\underline{\otimes}1)(k\mathbb{X}^i_{>w}\underline{\otimes}k\mathbb{X}^j) \subset [v]k\mathbb{X}^i_{>w}\underline{\otimes}k\mathbb{X}^j \subset k\mathbb{X}^{i+l(v)}_{>u}\underline{\otimes}k\mathbb{X}^j$ by definition of the lexicographical order. As $w{>}u$ (u is Lyndon) we have

$$(1\underline{\otimes}[v])([w]\underline{\otimes}1) = c([v]\underline{\otimes}[w]) \in k\mathbb{X}^{l(w)}_{\geq w}\underline{\otimes}k\mathbb{X}^{l(v)} \subset k\mathbb{X}^{l(w)}_{>u}\underline{\otimes}k\mathbb{X}^{l(v)}.$$

Furthermore $(1\underline{\otimes}[v])(k\mathbb{X}^l_{>w}\underline{\otimes}k\mathbb{X}^m) \subset k\mathbb{X}^l_{>w}\underline{\otimes}k\mathbb{X}^{i+l(v)} \subset k\mathbb{X}^l_{>u}\underline{\otimes}k\mathbb{X}^{i+l(v)}$ using the same argument. Finally for all $i, j, l, m \in \mathbb{N}$ with $i < l(v)$ we have $(k\mathbb{X}^i_{>v}\underline{\otimes}k\mathbb{X}^j)(k\mathbb{X}^l\underline{\otimes}k\mathbb{X}^m) \subset k\mathbb{X}^i_{>v}k\mathbb{X}^l\underline{\otimes}k\mathbb{X}^{j+m} \subset k\mathbb{X}^{i+l}_{>u}\underline{\otimes}k\mathbb{X}^{j+m}$, because if $a \in \mathbb{X}$ is shorter than v and $a > v$, then for all $b \in \mathbb{X}$ also $ab > vb$.
On the other hand

$$\begin{aligned}
\Delta([w])\Delta([v]) &\in \left([w]\underline{\otimes}1 + 1\underline{\otimes}[w] + \sum_{\substack{i+j=l(w)\\ i,j\neq 0}} k\mathbb{X}^i_{>w}\underline{\otimes}k\mathbb{X}^j\right)\cdot \\
&\quad \left([v]\underline{\otimes}1 + 1\underline{\otimes}[v] + \sum_{\substack{l+m=l(v)\\ l,m\neq 0}} k\mathbb{X}^l_{>v}\underline{\otimes}k\mathbb{X}^m\right) \\
&\subset [w][v]\underline{\otimes}1 + d([w]\underline{\otimes}[v]) + 1\underline{\otimes}[w][v] + \sum_{\substack{i+j=n\\ i,j\neq 0}} k\mathbb{X}^i_{>u}\underline{\otimes}k\mathbb{X}^j
\end{aligned}$$

using $([w]\underline{\otimes}1)(1\underline{\otimes}[v]) = [w]\underline{\otimes}[v] \in k\mathbb{X}^{l(w)}_{\geq w}\underline{\otimes}k\mathbb{X}^{l(v)} \subset k\mathbb{X}^{l(w)}_{>u}\underline{\otimes}k\mathbb{X}^{l(v)}$, again because $w>u$. Furthermore

$$([w]\underline{\otimes}1)(k\mathbb{X}^i_{>v}\underline{\otimes}k\mathbb{X}^j) \subset [w]k\mathbb{X}^i_{>v}\underline{\otimes}k\mathbb{X}^j \subset k\mathbb{X}^{l(w)+i}_{>wv}\underline{\otimes}k\mathbb{X}^j \subset k\mathbb{X}^{i+l(w)}_{>u}\underline{\otimes}k\mathbb{X}^j,$$

because u is Lyndon and thus $wv>vw = u$. Then we use $(1\underline{\otimes}[w])([v]\underline{\otimes}1) = c([w]\underline{\otimes}[v]) \in d([w]\underline{\otimes}[v]) + k\mathbb{X}^{l(w)}_{>w}\underline{\otimes}k\mathbb{X}^{l(v)} \subset d([w]\underline{\otimes}[v]) + k\mathbb{X}^{l(w)}_{>u}\underline{\otimes}k\mathbb{X}^{l(v)}$. If $i < l(v)$ then $k\mathbb{X}_{>w}, k\mathbb{X}^i_{>v} \subset k\mathbb{X}_{>u}$. Finally, $k\mathbb{X}_{>u}\underline{\otimes}k\mathbb{X}$ is a right ideal in $k\mathbb{X}\underline{\otimes}k\mathbb{X}$. As $d([w]\underline{\otimes}[v]) = \gamma_{w,v}[v]\underline{\otimes}[w]$ by Lemma 2.3.3 we obtain

$$\begin{aligned}\Delta([u]) &= \Delta([v])\Delta([w]) - \gamma^{-1}_{w,v}\Delta([w])\Delta([v]) \\ &\in 1\underline{\otimes}[u] + [u]\underline{\otimes}1 + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{>u}\underline{\otimes}k\mathbb{X}^j.\end{aligned}$$

□

To describe the comultiplication on arbitrary words we need an other subset of $\mathbb{X}$.

Definition 2.3.5. For $u, v \in \mathbb{X}$, u a Lyndon word we write $v \gg u$ if u is smaller than the first Lyndon letter of v. Furthermore

$$\mathbb{X}_{\gg u} := \{v \in \mathbb{X} | v \gg u\}, \mathbb{X}^m_{\gg u} := \mathbb{X}^m \cap \mathbb{X}_{\gg u}.$$

This subset is an important tool for the step from the setting of diagonal braidings to that of triangular braidings. It cannot be found in the work of Kharchenko. First we collect some auxiliary statements.

Remark 2.3.6. Let $u, v \in \mathbb{X}$, u a Lyndon word. Then:

1. $\mathbb{X}_{\gg u} = \{v \in \mathbb{X} |$ for all $i \in \mathbb{N} : v>u^i\}$.
2. If also v is Lyndon, $v>u$ implies $v \gg u$.
3. If $v \gg u$, then $\mathbb{X}_{\geq v} \subset \mathbb{X}_{\gg u}$.
4. If v is Lyndon, $v>u$ then $\mathbb{X}_{\geq v}\mathbb{X} \subset \mathbb{X}_{\gg u}$.
5. If v is Lyndon, $v>u$ then $\mathbb{X}_{\gg v} \subset \mathbb{X}_{\gg u}$.
6. If $v>u$ and $l(v) \leq l(u)$, then $v \gg u$.
7. $\mathbb{X}_{\gg u}\mathbb{X} \subset \mathbb{X}_{\gg u}$.
8. If $i \in \mathbb{N}$, then $\mathbb{X}^{il(u)}_{\geq u^i}\mathbb{X}_{\gg u} \subset \mathbb{X}_{\gg u}$.

Proof. For part 1 assume first that $v \gg u$ and let $v = v_1 \dots v_r$ be the Lyndon decomposition of v. This means that $v_1 > u$ by assumption and for all $i \in \mathbb{N}$ we obtain $v = v_1 \dots v_r > u^i$ by comparing the Lyndon letters lexicographically (Lemma 2.1.9), keeping in mind that the Lyndon decomposition of u^i consist of i Lyndon letters u. On the other hand let $v \in \mathbb{X}$ with $v > u^i$ for all $i \in \mathbb{N}$. Again let $v = v_1 \dots v_r$ bet the Lyndon decomposition. Because of $v > u$ we have $v_1 \geq u$. Assume $v_1 = u$. Then we find $i \in \mathbb{N}$ such that $v_1 = \dots = v_i = u$ and $v_{i+1}, \dots, v_r < u$. If $i = r$ we have $v = u^r < u^{r+1}$, a contradiction. If $i < r$ we have $v = u^i v_{i+1} \dots v_r < u^{i+1}$ by comparing the Lyndon letters - again a contradiction. Thus $v_1 > u$ and $v \gg u$.
Part 2 follows from the definition. Part 3: Let $w \in \mathbb{X}, w \geq v$. Then $w \geq v > u^i$ for all $i \in \mathbb{N}$ and so $w \in \mathbb{X}_{\gg u}$. For part 4 consider $a, b \in \mathbb{X}$ with $a \geq v$. Then $ab \geq a \geq v > u^i$ for all $i \in \mathbb{N}$. Part 5 is trivial.
Part 6: If v is Lyndon, this is part 2. Otherwise let $v = v_1 \dots v_r$ be the Lyndon decomposition of v. Then $v_1 \geq u$ and $l(v_1) < l(v) \leq l(u)$. So $v_1 > u$ and $v \gg u$.
For part 7 let $a \in \mathbb{X}_{\gg u}, b \in \mathbb{X}$. Let c be the first Lyndon letter of a. So we have $a \geq c > u$ and $ab \in \mathbb{X}_{\geq c}\mathbb{X} \subset \mathbb{X}_{\gg u}$ by part 4. Part 8: Assume $a \geq u^i, l(a) = il(u) = l(u^i), b \gg u$. First assume $j \geq i$. Then $ab \geq u^i b$ (because a and u^i have the same length) and $b > u^{j-i}$. Together this means $ab \geq u^i b > u^i u^{j-i} = u^j$. Now assume $j < i$. If u^j is the beginning of a, then also of ab and thus $ab > u^j$. If otherwise u^j is not the beginning of a, then $u^j < a$ (because $u^i \leq a$) implies $u^j < ab$. In any case $ab > u^j$ and thus $ab \gg u$. □

Remark 2.3.7. Let $u \in \mathbb{X}$ be a Lyndon word and $p, q \in \mathbb{N}$. Then

$$c(k\mathbb{X}^p \underline{\otimes} k\mathbb{X}^q_{\gg u}) \subset k\mathbb{X}^q_{\gg u} \underline{\otimes} k\mathbb{X}^p.$$

Proof. Let $v, w \in \mathbb{X}, v \gg u, l(v) = q, l(w) = p$. Then

$$c(w \underline{\otimes} v) \in k\mathbb{X}^q_{\geq v} \underline{\otimes} k\mathbb{X}^p \subset k\mathbb{X}^q_{\gg u} \underline{\otimes} k\mathbb{X}^p.$$

The last inclusion is by part 3 of the preceding remark. □

Corollary 2.3.8. Let $u \in \mathbb{X}$ be a Lyndon word, $n = l(u)$. Then

$$\Delta([u]) \in [u] \underline{\otimes} 1 + 1 \underline{\otimes} [u] + \sum_{\substack{i+j=n \\ i,j \neq 0}} k\mathbb{X}^i_{\gg u} \underline{\otimes} k\mathbb{X}^j.$$

Proof. Since Δ is graded we obtain with Lemma 2.3.4

$$\Delta([u]) \in [u] \underline{\otimes} 1 + 1 \underline{\otimes} [u] + \sum_{\substack{i+j=n \\ i,j \neq 0}} k\mathbb{X}^i_{> u} \underline{\otimes} k\mathbb{X}^j,$$

but for $i < l(u) = n$ we have by part 6 of Remark 2.3.6 $k\mathbb{X}^i_{> u} \subset k\mathbb{X}^i_{\gg u}$. □

Thus up to terms of a special form (simple tensors whose left tensorand is made up of monomials having a first Lyndon letter bigger than u) the $[u]$ behave like primitive elements. The aim of this section is to extend this observation to arbitrary words. In this sense the next two lemmas are generalizations of calculations used in [20] to our situation. The more general context asks for a more careful formulation of statements and proofs.

Lemma 2.3.9. Let $v \in \mathbb{X}$ be a Lyndon word, $r \in \mathbb{N}$ and $n := l(v^r)$. Then

$$\Delta([v^r]) \in \sum_{i=0}^{r} \binom{r}{i}_{\gamma_{v,v}} [v]^i \underline{\otimes} [v]^{r-i} + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{\gg v} \underline{\otimes} k\mathbb{X}^j.$$

Proof. We use induction on r. The case $r=1$ is the preceding corollary. So assume $r>1$. Then

$$\begin{aligned}
\Delta([v^r]) &= \Delta([v]^{r-1})\Delta([v]) \\
&\in \left(\sum_{i=0}^{r-1} \binom{r-1}{i}_{\gamma_{v,v}} [v]^i \underline{\otimes} [v]^{r-1-i} + \sum_{\substack{i+j=n-l(v) \\ i,j\neq 0}} k\mathbb{X}^i_{\gg v} \underline{\otimes} k\mathbb{X}^j \right) \\
&\quad \left([v]\underline{\otimes} 1 + 1 \underline{\otimes} [v] + \sum_{\substack{l+m=l(v) \\ l,m\neq 0}} k\mathbb{X}^l_{\gg v} \underline{\otimes} k\mathbb{X}^m \right).
\end{aligned}$$

Now note that $[v]^i k\mathbb{X}_{\gg v} \subset k\mathbb{X}_{\gg v}$ by part 8 of Remark 2.3.6 and that thus $k\mathbb{X}_{\gg v}\underline{\otimes} k\mathbb{X}$ is stable under left multiplication with elements from $[v]^i \underline{\otimes} k\mathbb{X}$. Furthermore $k\mathbb{X}_{\gg v}\underline{\otimes} k\mathbb{X}$ is a right ideal in $k\mathbb{X}\underline{\otimes} k\mathbb{X}$. As (again by part 8 of Remark 2.3.6)

$$\begin{aligned}
([v]^i \underline{\otimes} [v]^j)([v]\underline{\otimes} 1) &= \gamma^j_{v,v}[v]^{i+1}\underline{\otimes}[v]^j + k\mathbb{X}^{l(v^{i+1})}_{>v^{i+1}} \underline{\otimes} k\mathbb{X}^{l(v^k)} \\
&\subset \gamma^j_{v,v}[v]^{i+1}\underline{\otimes}[v]^j + k\mathbb{X}^{l(v^{i+1})}_{\gg v} \underline{\otimes} k\mathbb{X}^{l(v^k)},
\end{aligned}$$

this implies

$$\begin{aligned}
\Delta([v^r]) &\in \sum_{i=0}^{r-1} \binom{r-1}{i}_{\gamma_{v,v}} \left(\gamma^i_{v,v}[v]^{i+1}\underline{\otimes}[v]^{r-1-i} + [v]^i \underline{\otimes} [v]^{r-i} \right) + \\
&\quad + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{\gg v} \underline{\otimes} k\mathbb{X}^j.
\end{aligned}$$

Using the recursion formula for the q-binomial coefficients we obtain the claim. $\square$

Lemma 2.3.10. Let $u \in \mathbb{X}$ and $u = u_1 \dots u_t v^r, u_1 \geq \dots \geq u_t > v$ be the Lyndon decomposition with $r, t \geq 1$. Define $z := u_1 \dots u_t, n := l(u)$. Then

$$\Delta([u]) \in [u]\underline{\otimes}1 + \sum_{i=0}^{r} \binom{r}{i}_{\gamma_{v,v}} \gamma_{z,v}^i [v]^i \underline{\otimes}[z][v]^{r-i} + \sum_{\substack{i+j=n\\ i,j\neq 0}} k\mathbb{X}_{\gg v}^i \underline{\otimes} k\mathbb{X}^j.$$

Proof. We use induction on t. First assume $t = 1$. Then z is Lyndon and $z>v$. By Part 5 of Remark 2.3.6 we have $k\mathbb{X}_{\gg z} \subset k\mathbb{X}_{\gg v}$. So we obtain using the preceding lemma

$$\begin{aligned}
\Delta([zv^r]) &= \Delta([z])\Delta([v]^r) \\
&\in \left([z]\underline{\otimes}1 + 1\underline{\otimes}[z] + \sum_{\substack{i+j=l(z)\\ i,j\neq 0}} k\mathbb{X}_{\gg v}^i \underline{\otimes} k\mathbb{X}^j\right) \\
&\quad \left(\sum_{i=0}^{r} \binom{r}{i}_{\gamma_{v,v}} [v]^i \underline{\otimes}[v]^{r-i} + \sum_{\substack{l+m=l(v^r)\\ l,m\neq 0}} k\mathbb{X}_{\gg v}^l \underline{\otimes} k\mathbb{X}^m\right).
\end{aligned}$$

Again $k\mathbb{X}_{\gg v}\underline{\otimes}k\mathbb{X}$ is a right ideal and stable under left multiplication with $1\underline{\otimes}[z]$.
Moreover by Part 8 of Remark 2.3.6 $(1\underline{\otimes}[z])([v]^i\underline{\otimes}[v]^{r-i}) \in \gamma_{z,v}^i[v]^i\underline{\otimes}[z][v]^{r-i} + k\mathbb{X}_{>v^i}^{l(v^i)}\underline{\otimes}k\mathbb{X}^{n-l(v^i)} \subset \gamma_{z,v}^i[v]^i\underline{\otimes}[z][v]^{r-i} + k\mathbb{X}_{\gg v}^{l(v^i)}\underline{\otimes}k\mathbb{X}^{n-l(v^i)}$. As furthermore $[z] \in k\mathbb{X}_{\geq z} \subset k\mathbb{X}_{\gg v}$ (Part 3 of Remark 2.3.6) we obtain

$$\Delta([u][v^r]) \in [u][v]^r\underline{\otimes}1 + \sum_{i=0}^{r} \binom{r}{i}_{\gamma_{v,v}} \gamma_{z,v}^i [v]^i \underline{\otimes}[z][v]^{r-i} + \sum_{\substack{i+j=n\\ i,j\neq 0}} k\mathbb{X}_{\gg v}^i \underline{\otimes} k\mathbb{X}^j.$$

So assume now $t > 1$ and let $w := u_2 \dots u_t$. The induction hypothesis and the preceding lemma give

$$\begin{aligned}
\Delta([u_1]) &\in [u_1]\underline{\otimes}1 + 1\underline{\otimes}[u_1] + \sum_{\substack{i+j=l(u)\\ i,j\neq 0}} k\mathbb{X}_{\gg v}^i \underline{\otimes} k\mathbb{X}^j \\
\Delta([z][v]^r) &\in [z][v^r]\underline{\otimes}1 + \sum_{i=0}^{r} \binom{r}{i}_{\gamma_{v,v}} \gamma_{w,v}^i [v]^i \underline{\otimes}[w][v]^{r-i} \\
&\quad + \sum_{\substack{l+m=l(v^r)\\ l,m\neq 0}} k\mathbb{X}_{\gg v}^l \underline{\otimes} k\mathbb{X}^m.
\end{aligned}$$

Multiplying these we use that $k\mathbb{X}_{\gg v}\underline{\otimes}k\mathbb{X}$ is a right ideal and stable under left multiplication with elements from $k\mathbb{X}_{>v}\underline{\otimes}k\mathbb{X}$ and $1\underline{\otimes}k\mathbb{X}$. Note $[u_1] \in k\mathbb{X}_{\gg v}$, $[z][v^r] \in k\mathbb{X}_{\gg v}$. Together with

$$(1\underline{\otimes}[u_1])([v]^i\underline{\otimes}[z][v]^{r-i}) \in \gamma^i_{u_1,v}[v]^i\underline{\otimes}[u_1][z][v]^{r-i} + k\mathbb{X}^{l(v^i)}_{\gg v}\underline{\otimes}k\mathbb{X}^{n-l(v^i)}$$

we have

$$\begin{aligned}\Delta([u_1][w][v^r]) \quad &\in \quad [u_1][w][v^r]\underline{\otimes}1 + \sum_{i=0}^{r}\binom{r}{i}_{\gamma_{v,v}}\gamma^i_{w,v}\gamma^i_{u_1,v}[v]^i\underline{\otimes}[u_1][w][v]^{r-i} \\ &\quad + \sum_{\substack{i+j=n\\ i,j\neq 0}} k\mathbb{X}^i_{\gg v}\underline{\otimes}k\mathbb{X}^j,\end{aligned}$$

establishing the claim. □

2.3.3 The PBW basis

Now we will apply the results from Subsections 2.3.1 and 2.3.2 to prove the Theorems 2.2.4 and 2.2.6. Again fix a finite-dimensional braided vector space (V, c), assume that it is left triangular with respect to the basis X and denote by d the diagonal component of c. Abbreviate $[-] := [-]_{d^{-1}}$. Let $I \subsetneq k\mathbb{X}$ be a braided biideal and let $S := S_I$ be the set of PBW generators from Definition 2.2.3, denote the corresponding height function by h and the PBW set generated by this data by $B := B(S_I, <, h)$. We will need some further notation. For $n \in \mathbb{N}$ and a Lyndon word $v \in \mathbb{X}$ define $B^n := B \cap \mathbb{X}^n$, $B_{\gg v} := B \cap \mathbb{X}_{\gg v}$ and $B^n_{\gg v} := B^n \cap B_{\gg v}$.
The next proposition collects some statements which will be useful in the sequel. Analogues of parts 1,3 and 4 are also used in [20].

Proposition 2.3.11. Let r be an admissible endomorphism of $V \otimes V$. For every $m \in \mathbb{N}_0, u, v \in \mathbb{X}$, v a Lyndon word we have the following inclusions

1. $k\mathbb{X}_{\succeq u} \subset k[B_{\succeq u}]_r + I$,
2. $k\mathbb{X}^m_{\gg v} \subset k[B^m_{\gg v}]_r + \sum\limits_{0\le i<m} k[B^i]_r + I$,
3. $k\mathbb{X}^m \subset \sum\limits_{0\le i\le m} k[B^i]_r + I$,
4. $k\mathbb{X} = k[B]_r + I$.

Proof. First note that for $x, y, a, b \in \mathbb{X}$ $a \succ b$ implies $xay \succ xby$. For part 1 we proceed by downward induction along the standard order (this works because the standard order satisfies the ascending chain condition). For $u = 1$ the inclusion is valid. Now assume $u \prec 1$ and that for all words $\succ u$ the inclusion is valid. Let $w \succeq u, m := l(w)$. If $w \in B$ we have $w \in [w]_r + k\mathbb{X}_{\succ w} \subset k[B_{\succeq w}]_r + I$ by induction. Assume $w \notin B$ and let $w = w_1^{e_1} \dots w_t^{e_t}$ be the Lyndon decomposition of w. As $w \notin B$ we find an $0 \leq i \leq t$ such that either $w_i \notin S$ or $e_i \geq h(w_i)$. In the first case we have $w_i \in k\mathbb{X}_{\succ w_i} + I$, in the second case $w_i^{e_i} \in k\mathbb{X}_{\succ w_i^{e_i}} + I$. Anyway this implies $w \in k\mathbb{X}_{\succ w} + I$, but thus $[w]_r \in k\mathbb{X}_{\succ w} + I \subset k[B_{\succ w}]_r + I \subset k[B_{\succ u}]_r + I$ by induction. Now for part 2 assume $w \in k\mathbb{X}^m_{\gg v}$. Then by part 1 $w \in k\mathbb{X}_{\succeq w} \subset k[B_{\succeq w}]_r + I \subset k[B^m_{\succeq w}]_r + \sum\limits_{0 \leq i < m} k[B^i]_r + I$. In view of 2.3.6 we have $k\mathbb{X}^m_{\succeq w} \subset k\mathbb{X}^m_{\gg v}$. For part 3 let u_0 be the smallest word of degree m. For $u \in \mathbb{X}^m$ we have $u \succeq u_0$ and thus by part 1 $u \in k[B_{\succeq u_0}]_r + I$ and this is a subset of $\sum\limits_{0 \leq i \leq m} k[B^i]_r$. Finally Part 4 follows from Part 3. $\square$

For the rest of the proof we use the main ideas of [20], but in a different and more general setting. As the triangular braiding requires a more careful analysis we work in the tensor algebra rather than in the quotient of the tensor algebra by the ideal I (as Kharchenko does, not regarding the biproduct with the group algebra). This enables us to use linear maps as tools where Kharchenko argues by inspection of the occurring terms, a method for which our situation seems to be too complicated.

Lemma 2.3.12. Let r be an admissible endomorphism of $V \otimes V$. Then the set $[B_I]_r$ is linearly independent.

Proof. The map $[-]_r : k\mathbb{X} \to k\mathbb{X}$ is homogeneous. Furthermore it is surjective (use part 4 of the proposition above for the subspace (0) to see that $k\mathbb{X} = k[\mathbb{X}]_r$). As the homogeneous components are finite-dimensional, $[-]_r$ is bijective and maps the linearly independent set B_I onto $[B_I]_r$. So $[B_I]_r$ is linearly independent. $\square$

The next theorem is the key step to the final theorem combining the results of the preceding section on the comultiplication with the results of this section.

Theorem 2.3.13. Let $I \subsetneq k\mathbb{X}$ be a braided biideal in $k\mathbb{X}$. Then $[B_I]$ spans a k-linear complement of I.

Proof. By the proposition above all we need to show is that $k[B_I]$ and I have trivial intersection. For $n \geq 0$ let $U_n := k\text{-span}\{[u] | u \in B_I, l(u) \leq n\}$. We show by induction on n that for all $n \in \mathbb{N}$ we have $U_n \cap I = (0)$. First let

$n = 0$. Then $U_0 = k1$ and thus $U_0 \cap I = (0)$ since I is proper a ideal. Now assume $n > 0$. Assume $0 \neq T \in U_n \cap I$. So we can write T as a (finite) sum

$$T = \sum_{\substack{u \in B_I \\ l(u) \leq n}} \alpha_u [u].$$

We may assume that there is a $u \in B^n$ such that $\alpha_u \neq 0$. Now choose v as the (lexicographically) smallest Lyndon letter occurring in the Lyndon decomposition of words $u \in B_I^n$ with $\alpha_u \neq 0$. Because of the minimality of v, it occurs in Lyndon decompositions of words $u \in B_I^n$ with $\alpha_u \neq 0$ only at the end. Let t be the maximal number of occurrences of v in a Lyndon decomposition of word $u \in B_I^n$ with $\alpha_u \neq 0$. Thus we can decompose the sum for T in the following way

$$T = \sum_{u \in O} \alpha_u [a_u][v]^t + \sum_{u \in P} \alpha_u [a_u][v]^{t_u} + \sum_{u \in Q} \alpha_u [u] + \sum_{u \in R} \alpha_u [u],$$

where $O, P, Q, R \subset B_I$ and the words a_u for $u \in O \cup P$ are chosen such that

- O contains all words $u \in B_I^n$ of length n with $\alpha_u \neq 0$ such that the Lyndon decomposition of u ends with v^t. Furthermore $u = a_u v^t$.

- P contains all words $u \in B_I^n$ of length n with $\alpha_u \neq 0$ such that the Lyndon decomposition of u ends with v^{t_u} for some $0 \neq t_u < t$. Furthermore $u = a_u v^{t_u}$.

- Q contains all words $u \in B_I^n$ of length n with $\alpha_u \neq 0$ that do not have the Lyndon letter v in their Lyndon decomposition.

- R contains all words $u \in B_I^n$ of length less than n with $\alpha_u \neq 0$.

Note that for all $u \in O$ we have $a_u \neq 1$: $a_u = 1$ would imply that $u = v^l \in k\mathbb{X}_{\succ v^l}$, but as well $l < h_I(v)$ because $u \in B$ which is a contradiction to the definition of $h_I(v)$. By analyzing the four terms we will show

$$\begin{aligned} \Delta(T) \quad \in \quad & T \underline{\otimes} 1 + [v]^t \underline{\otimes} \sum_{u \in O} \alpha_u \gamma_{a_u, v}^t [a_u] + \sum_{i=0}^{t-1} [v]^i \underline{\otimes} k\mathbb{X}^{n - l(v^i)} + \\ & \sum_{\substack{i+j=n \\ i,j \neq n}} k[B_{\gg v}^i] \underline{\otimes} k\mathbb{X}^j + \sum_{i+j<n} k[B^i] \underline{\otimes} k\mathbb{X}^j + I \underline{\otimes} k\mathbb{X}. \end{aligned}$$

First consider $u \in O, l := t$ or $u \in P, l := t_u < t$. Note that then $a_u \neq 1$. We obtain

$$\begin{aligned}
\Delta([a_u][v]^l) &\in [a_u][v]^l \underline{\otimes} 1 + \sum_{i=0}^{l} \binom{l}{i}_{\gamma_{v,v}} \gamma^i_{a_u,v} [v]^i \underline{\otimes} [a_u][v]^{l-i} + \\
&\quad + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{\gg v} \underline{\otimes} k\mathbb{X}^j \\
&\subset [a_u][v]^l \underline{\otimes} 1 + [v]^l \underline{\otimes} \gamma^l_{a_u,v}[a_u] + \sum_{i=0}^{l-1} [v]^i \underline{\otimes} k\mathbb{X}^{n-l(v^i)} + \\
&\quad \sum_{\substack{i+j=n \\ i,j\neq 0}} k[B^i_{\gg v}] \underline{\otimes} k\mathbb{X}^j + \sum_{i+j<n} k[B^i] \underline{\otimes} k\mathbb{X}^j + I \underline{\otimes} k\mathbb{X}.
\end{aligned}$$

In both cases ($u \in O$ or $u \in P$) this delivers the right terms in the sum for the word u. Now consider $u \in Q$ and let w be the largest Lyndon letter occurring in the Lyndon decomposition of u. Let $u = l_1 \dots l_\mu w^l$ be the Lyndon decomposition of u and define $a := l_1 \dots l_\mu$. Then $u = aw^l$ and $w > v$ by construction of v. This leads to

$$\begin{aligned}
\Delta([u]) &\in [u] \underline{\otimes} 1 + \sum_{i=0}^{l} \binom{l}{i}_{\gamma_{w,w}} \gamma^i_{a,w} [w]^i \underline{\otimes} [a][w]^{l-i} + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{\gg w} \underline{\otimes} k\mathbb{X}^j \\
&\subset [u] \underline{\otimes} 1 + \sum_{\substack{i+j=n \\ i,j\neq 0}} k\mathbb{X}^i_{\gg v} \underline{\otimes} k\mathbb{X}^j \\
&\subset [u] \underline{\otimes} 1 + \sum_{\substack{i+j=n \\ i,j\neq 0}} k[B^i_{\gg v}] \underline{\otimes} k\mathbb{X}^j + \sum_{i+j<n} k[B^i] \underline{\otimes} k\mathbb{X}^j + I \underline{\otimes} k\mathbb{X}.
\end{aligned}$$

Finally consider $u \in R$. Then $l(u) < n$ and we obtain

$$\begin{aligned}
\Delta([u]) &\in [u] \underline{\otimes} 1 + \sum_{i+j<n} k\mathbb{X}^i \underline{\otimes} k\mathbb{X}^j \\
&\subset [u] \underline{\otimes} 1 + \sum_{i+j<n} k[B^i] \underline{\otimes} k\mathbb{X}^j + I \underline{\otimes} k\mathbb{X}.
\end{aligned}$$

Now by induction assumption we find a $\phi \in (k\mathbb{X})^*$ such that

$$\phi(I) = 0, \phi([v]^t) = 1 \text{ and } \forall u \in B \setminus \{v^t\} \text{ with } l(u) < n : \phi([u]) = 0.$$

With the inclusion we showed above we get

$$\begin{aligned}(\phi\underline{\otimes}\,\mathrm{id})\Delta(T) &\in \sum_{u\in O}\alpha_u\gamma^t_{a_u,v}[a_u] + \sum_{\substack{i+j=n\\ i,j\neq 0}}\phi(k[B^i_{\gg v}])k\mathbb{X}^j + \\ &\quad \sum_{i+j<n}\phi(k[B^i])k\mathbb{X}^j + \phi(I)k\mathbb{X} \\ &\subset \sum_{u\in O}\alpha_u\gamma^t_{a_u,v}[a_u] + 0 + \sum_{j<n-tl(v)} k\mathbb{X}^j + 0 \\ &\subset \{k[B^{n-tl(v)}] \oplus U_{n-tl(v)-1}) \setminus \{0\} \subset U_{n-1}\setminus\{0\}.\end{aligned}$$

Note that we cannot obtain 0, because we have a non-zero component in degree $n - tl(v)$. On the other hand, as I is a biideal, we have

$$(\phi\underline{\otimes}\,\mathrm{id})\Delta(T) \in \phi(I)k\mathbb{X} + \phi(k\mathbb{X})I \subset I.$$

Thus $(\phi\underline{\otimes}\,\mathrm{id})\Delta(T) \in I \cap (U_{n-1}\setminus\{0\})$, but by induction assumption this is the empty set, a contradiction. ☐

Corollary 2.3.14. Let $I \subsetneq k\mathbb{X}$ be a braided biideal and r an admissible endomorphism of $V\otimes V$. Then $[B_I]_r$ spans a k-linear complement of I.

Proof. Again all we have to show is that $k[B]_r \cap I = \{0\}$. Assume $0 \neq T \in k[B]_r \cap I$. We can write T as

$$T = \alpha[u]_r + \sum_{w\succ u}\beta_w[w]_r$$

with $\alpha \neq 0$. Then by the Proposition 2.3.11 and Lemma 2.3.3 we obtain first

$$T \in \alpha u + k\mathbb{X}_{\succ u}$$

and from this

$$T \in \alpha[u] + k\mathbb{X}_{\succ u} \subset \alpha[u] + k[B_{\succ u}] + I.$$

Write $T = \alpha[u] + x + i$ with $x \in k[B_{\succ u}], i \in I$. Now by the theorem above we obtain $\alpha[u] + x \in I \cap k[B] = \{0\}$. But $[B]$ is linearly independent, which implies $\alpha = 0$, a contradiction. ☐

Now Theorem 2.2.4 follows as a special case of the following remark.

Remark 2.3.15. Let $I \subsetneq k\mathbb{X}$ be a braided biideal, r an admissible endomorphism of $V\otimes V$ and $\pi : k\mathbb{X} \to k\mathbb{X}/I$ the quotient map. Then $\pi([B_I]_r)$ is a basis of $(k\mathbb{X})/I$.

Proof. By Corollary 2.3.14, $[B_I]_r$ is a basis for a complement of I. As $I = \ker\pi$, π induces a k-linear isomorphism $k[B_I]_r \to k\mathbb{X}/I$, mapping the basis $[B_I]_r$ into $\pi([B_I]_r)$. This proves the remark. Theorem 2.2.4 follows by using $r = 0$ and $r = c$. $\square$

Now we can also prove the result on the height function.

Proof of Theorem 2.2.6. Let $n := l(v^h)$. We have an element of I of the form

$$T := [v]^h + \sum_{\substack{u>v^h\\ l(u)=n}} \alpha_u[u] + \sum_{l(w)<n} \alpha_w[w] \in I.$$

For every $u \in \mathbb{X}^n$ with $u>v^h$ we have $u\gg v$ and thus we obtain for the coproduct using Lemmas 2.3.9 and 2.3.10

$$\begin{aligned}
\Delta(T) \quad \in \quad & T\underline{\otimes}1 + \sum_{i=0}^{h-1}\binom{h}{i}_{\gamma_{v,v}} [v]^i\underline{\otimes}[v]^{h-i} + \sum_{\substack{i+j=n\\ j\neq 0}} k\mathbb{X}^i_{\gg v}\underline{\otimes}k\mathbb{X}^j + \\
& + \sum_{\substack{i+j<n\\ j\neq 0}} k\mathbb{X}^i\underline{\otimes}k\mathbb{X}^j \\
\subset \quad & T\underline{\otimes}1 + \sum_{i=0}^{h-1}\binom{h}{i}_{\gamma_{v,v}} [v]^i\underline{\otimes}[v]^{h-i} + \sum_{i+j=n} k[B^i_{\gg v}]\underline{\otimes}k[B^j] + \\
& \sum_{i+j<n} k[B^i]\underline{\otimes}k[B^j] + I\underline{\otimes}k\mathbb{X} + k\mathbb{X}\underline{\otimes}I.
\end{aligned}$$

Now because of $k\mathbb{X}\underline{\otimes}k\mathbb{X} = (I\underline{\otimes}k\mathbb{X} + k\mathbb{X}\underline{\otimes}I) \oplus (k[B]\underline{\otimes}k[B])$ we can construct a k-linear map $\phi_1 : k\mathbb{X}\otimes k\mathbb{X} \to k$ such that

$$\begin{aligned}
&\forall (b,b') \in (B\times B)\setminus\{(v,v^{h-1})\} : \phi_1([b]\underline{\otimes}[b']) = 0,\\
&\phi_1([v]\underline{\otimes}[v]^{h-1}) = 1,\\
&\phi_1(I\underline{\otimes}k\mathbb{X} + k\mathbb{X}\underline{\otimes}I) = 0.
\end{aligned}$$

As $T \in I$ we have $\phi_1\Delta(T) = 0$ and on the other hand using what we proved above

$$0 = \phi_1\Delta(T) = \binom{h}{h-1}_{\gamma_{v,v}} = 1 + \gamma_{v,v} + \ldots + {\gamma_{v,v}}^{h-1}.$$

This shows that $\gamma_{v,v}$ is a root of unity, say of order t (set $t = 1$ if $\gamma_{v,v} = 1$). Let $p := \operatorname{char} k$ and define q by

$$q := \begin{cases} p & \text{if} \quad p > 0,\\ 1 & \text{if} \quad p = 0.\end{cases}$$

Now we can write $h = tq^l a$ with $a, l \in \mathbb{N}$. If $q \neq 1$ we may assume that q does not divide a. We want to show that $a = 1$. So assume $a > 1$. In this case we can construct a k-linear map $\phi_2 : k\mathbb{X} \otimes k\mathbb{X} \to k$ with

$$\begin{aligned}
&\forall (b, b') \in (B \times B) \setminus \{(v^{tq^l}, v^{tq^l(a-1)})\} : \phi_2([b]\underline{\otimes}[b']) = 0, \\
&\phi_2([v]^{tq^l}\underline{\otimes}[v]^{tq^l(a-1)}) = 1, \\
&\phi_2(I\underline{\otimes}k\mathbb{X} + k\mathbb{X}\underline{\otimes}I) = 0.
\end{aligned}$$

Using that $\gamma_{v,v}$ is a primitive t-th root of unity (resp. $\gamma_{v,v} = 1$ and $t = 1$) we obtain that in k

$$0 = \phi_2\Delta(T) = \binom{tq^l a}{tq^l}_{\gamma_{v,v}} = \binom{q^l a}{q^l} = \binom{a}{1} = a.$$

This is a contradiction to the assumptions we made on a. Thus $h = tq^l$. In particular if $\operatorname{char} k = 0$, then $q = 1$ and because $t = h > 1$ we obtain $\gamma_{v,v} \neq 1$. □

2.4 Transfer to right triangular braidings

In principle one could do a similar proof as above for right triangular braidings, but an easy argument shows that the right triangular case follows from the left triangular case. Obviously c is a right triangular braiding if and only if $\tau c \tau$ is left triangular, where τ denotes the usual flip map. The key observation is

Proposition 2.4.1. Let $(R, \mu, \eta, \Delta, \varepsilon, c)$ be a braided bialgebra.
Then also $R^{op,cop} := (R, \mu\tau, \eta, \tau\Delta, \varepsilon, \tau c \tau)$ is a braided bialgebra.

Proof. Let $\mu^{op} := \mu\tau$ and $\Delta^{cop} := \tau\Delta$. Of course (R, μ^{op}, η) is an algebra, $(R, \Delta^{cop}, \varepsilon)$ is a coalgebra and $(R, \tau c \tau)$ is a braided vector space. Checking the compatibility of $\mu^{op}, \eta, \Delta^{cop}, \varepsilon$ with $\tau c \tau$ is tedious. We will do one example, namely the calculation that $\tau c \tau \circ (\mu^{op} \otimes R) = (R \otimes \mu^{op})(\tau c \tau \otimes R)(R \otimes \tau c \tau)$.

We calculate

$$
\begin{aligned}
&(R\otimes\mu^{op})(\tau c\tau\otimes R)(R\otimes\tau c\tau) =\\
&(R\otimes\mu)\underline{(R\otimes\tau)(\tau\otimes R)(R\otimes\tau)}(R\otimes\tau)(c\tau\otimes R)(R\otimes\tau c\tau) =\\
&\underline{(R\otimes\mu)(\tau\otimes R)(R\otimes\tau)}\circ\underline{(\tau\otimes R)(R\otimes\tau)(c\otimes R)}(\tau\otimes R)(R\otimes\tau c\tau) =\\
&\tau(\mu\otimes R)\circ(R\otimes c)\underline{(\tau\otimes R)(R\otimes\tau)(\tau\otimes R)}(R\otimes\tau c\tau) =\\
&\tau(\mu\otimes R)(R\otimes c)\underline{(R\otimes\tau)(\tau\otimes R)(R\otimes c)}(R\otimes\tau) =\\
&\tau\underline{(\mu\otimes R)(R\otimes c)(c\otimes R)}\circ\underline{(R\otimes\tau)(\tau\otimes R)(R\otimes\tau)} =\\
&\tau c\underline{(R\otimes\mu)\circ(\tau\otimes R)(R\otimes\tau)}(\tau\otimes R) =\\
&\tau c\tau(\mu\otimes R)(\tau\otimes R) =\\
&\tau c\tau(\mu^{op}\otimes R),
\end{aligned}
$$

where we use (in this order): $\tau^2=\mathrm{id}_{V\otimes V}$, the braid equation for τ, μ, c commute with τ, again the braid equation for τ and $\tau^2=\mathrm{id}_{V\otimes V}$, c commutes with τ, μ commutes with c and the braid equation for τ and finally again that μ commutes with τ. The other calculations work similarly (use graphical calculus as a tool for intuition).
Finally we have to check that $\Delta: R\to R\underline{\otimes}R$ and $\varepsilon: R\to k$ are algebra morphisms, where $R\underline{\otimes}R$ is an algebra with multiplication $(\mu\otimes\mu)(R\otimes\tau c\tau\otimes R)$. For ε this is trivial. For Δ we have to check

$$\Delta^{cop}\mu^{op}=(\mu^{op}\otimes\mu^{op})(R\otimes\tau c\tau\otimes R)(\Delta^{cop}\otimes\Delta^{cop}).$$

As R is a braided bialgebra the left hand side is

$$\tau\Delta\mu\tau=\tau(\mu\otimes\mu)(R\otimes c\otimes R)(\Delta\otimes\Delta)\tau.$$

Now because Δ,μ commute with τ this is equal to

$$(\mu\otimes\mu)(R\otimes\tau\otimes R)(\tau\otimes\tau)(R\otimes\tau c\tau\otimes R)(\tau\otimes\tau)(R\otimes\tau\otimes R)(\Delta\otimes\Delta).$$

Thus it suffices to show

$$(R\otimes\tau\otimes R)(\tau\otimes\tau)(R\otimes\tau c\tau\otimes R)(\tau\otimes\tau)(R\otimes\tau\otimes R)=(\tau\otimes\tau)(R\otimes\tau c\tau\otimes R)(\tau\otimes\tau),$$

but this is trivial (check on elements). □

Related material can be found in [2].
Assume now that (V,c) is a braided vector space. Denote the braided tensor bialgebra defined in Section 1.4 by $(T(V,c),\mu,\eta,\Delta_c,\varepsilon,c)$. As an algebra this is $T(V)$.

Proposition 2.4.2. Let (V, c) be a braided vector space. Let

$$\phi : T(V, c) \to T(V, \tau c\tau)^{op,cop}$$

be the unique algebra morphism $T(V) \to T(V)^{op}$ given by $\phi|V = \mathrm{id}_V$. Then ϕ is an isomorphism of braided bialgebras.
For $v_1, \ldots, v_n \in V$ we have

$$\phi(v_1 \otimes \ldots \otimes v_n) = v_n \otimes \ldots \otimes v_1.$$

Proof. Lemma 1.4.15 gives us the existence of ϕ as a morphism of braided bialgebras because (V, c) is a braided subspace of $T(V, \tau c\tau)^{op,cop}$. By construction we see that the map ϕ has the form given in the proposition and that it is bijective. □

Now we can prove the existence of the PBW basis in the right triangular case.

Theorem 2.4.3. Assume that (V, c) is a finite-dimensional right triangular braided vector space and $I \subsetneq T(V, c)$ is a braided biideal. Then there is a totally ordered subset $S \subset T(V, c)$ and a height function $h : S \to \mathbb{N} \cup \{\infty\}$ such that the images of the PBW set generated by S and h form a basis of $T(V, c)/I$.
Let

$$\phi : T(V, c) \to T(V, \tau c\tau)^{op,cop}$$

be the isomorphism from Proposition 2.4.2. We have

$$S = \phi^{-1}(S_{\phi(I)}) \text{ and } h = h_{\phi(I)}\phi,$$

and the order on the set S is the opposite of the order on $S_{\phi(I)}$.

Proof. As $\phi(I)$ is a braided biideal in $T(V, \tau c\tau)^{op,cop}$ it is also a braided biideal in $T(V, \tau c\tau)$. As c is right triangular we have that $\tau c\tau$ is left triangular. So we find a set $S_{\phi(I)} \subset T(V, \tau c\tau)$ with a total ordering $<$ and a height function $h_{\phi(I)} : S \to \mathbb{N} \cup \{\infty\}$ such that the PBW set generated by these data in $T(V, \tau c\tau)$ is a basis for a complement of $\phi(I)$. The PBW set generated in $T(V, \tau c\tau)^{op,cop}$ by $S_{\phi(I)}$ with reversed order and height function $h_{\phi(I)}$ is the same set and thus also a basis for a complement of $\phi(I)$. The claim follows by transferring this set to $T(V, c)$ via ϕ^{-1}. □

We have the following nice Corollary of Proposition 2.4.2.

Corollary 2.4.4. Let (V, c) be a braided vector space. Then

$$\mathcal{B}(V, \tau c\tau) \simeq \mathcal{B}(V, c)^{op,cop}$$

as braided graded Hopf algebras.

Proof. Denote the braiding of the braided bialgebra $\mathcal{B}(V, c)$ by $\hat{c}$. By Proposition 2.4.1 $\mathcal{B}(V, c)^{op,cop}$ is a braided bialgebra with braiding $\tau\hat{c}\tau$. It is easy to check that $\mathcal{B}(V, c)^{op,cop}$ has the properties of the Nichols algebra of $(V, \tau c\tau)$ from Definition 1.4.7. □

2.5 Application to pointed Hopf algebras with abelian coradical

In this section we will show how to obtain a PBW basis for a Hopf algebra generated by an abelian group G and a finite-dimensional G-module spanned by skew primitive elements. On one hand this yields a generalization of the result in [20] as there the skew primitive elements are assumed to be semi-invariants (i.e. that the group acts on them by a character). On the other hand we lose some properties of the basis as already mentioned in Remark 2.2.5.
Let $A = \cup_{n\geq 0} A_n$ be a filtered algebra. We can define a map

$$\pi : A \to \operatorname{gr} A$$

by setting $\pi(0) := 0$ and for all $0 \neq a \in A : \pi(a) := a + A_{n-1}$ for the unique $n \geq 0$ such that $a \in A_n \setminus A_{n-1}$ (where $A_{-1} := \{0\}$ as usual). We will use this map to obtain PBW bases for A from homogeneous PBW bases of the associated graded algebra $\operatorname{gr} A$.

Proposition 2.5.1. Let $A = \cup_{n\geq 0} A_n$ be a filtered algebra and $(P, S, <, h)$ a PBW basis for $\operatorname{gr} A$ such that $P \subset \operatorname{gr} A(0) = A_0$ and S is made up of homogeneous elements. Then there is a PBW basis $(P, S', <', h')$ of A such that for all $a, b \in S'$

$$\pi(a) \in S, h'(a) = h(\pi(a)) \text{ and } a < b \Leftrightarrow \pi(a) <' \pi(b).$$

Proof. For all $s \in S \cap \operatorname{gr} A(n)$ we find $\hat{s} \in A_n \setminus A_{n-1}$ such that $\pi(\hat{s}) = s$. Define

$$S' := \{\hat{s} | s \in S\}.$$

The map $S \to S', s \mapsto \hat{s}$ is bijective. So we can transfer the height function h and the order $<$ to S' obtaining h' and $<'$.

Assume we have $b := s_1^{e_1} \dots s_r^{e_r} p \in B(P, S, <, h)$. We define a lift

$$\hat{b} := \hat{s_1}^{e_1} \dots \hat{s_r}^{e_r} p \in B(P, S', <', h').$$

As the s_i and p are homogeneous (say of degrees n_i and 0), also b is homogeneous, say of degree n. Then

$$\begin{aligned} b &= (\hat{s_1} + A_{n_1-1})^{e_1} \dots (\hat{s_r} + A_{n_r-1})^{e_r} (p + A_{-1}) \\ &= \hat{s_1}^{e_1} \dots \hat{s_r}^{e_r} p + A_{n-1} = \hat{b} + A_{n-1} \end{aligned}$$

in $\operatorname{gr} A(n) = A_n / A_{n-1}$. We have $\hat{b} \in A_n \setminus A_{n-1}$, because otherwise ($\hat{b} \in A_{n-1}$) we had $b = 0$, but this is an element of a basis.

Let $B_n := B(P, S, <, h) \cap \operatorname{gr} A(n)$ and $\hat{B}_n := \{\hat{b} | b \in B_n\}$. We will show by induction on $n \geq 0$ that $\hat{B}_0 \cup \ldots \cup \hat{B}_n$ generates A_n as a vector space. For the case $n = 0$ one has to check that P is a basis of $A_0 = \operatorname{gr} A(0)$, which is easy. Assume $n \geq 0$ and $a \in A_n \setminus A_{n-1}$. We have $\pi(a) = a + A_{n-1} \in \operatorname{gr} A(n)$ and thus $\pi(a)$ is a linear combination of elements of B_n i.e.

$$\pi(a) = a + A_{n-1} \in kB_n = \sum_{b \in B_n} k(\hat{b} + A_{n-1}).$$

So we get that a is a linear combination of elements from $\hat{B}_n$ and A_{n-1} and by induction assumption a is a linear combination of elements from $\hat{B}_0 \cup \ldots \cup \hat{B}_n$.

We are left to show that $B(P, S', <', h')$ is linearly independent. Assume we have for all $b \in B_n$ scalars $\alpha_b \in k$ such that

$$\sum_{b \in B_n} \alpha_b \hat{b} \in A_{n-1}.$$

It suffices to show that $\alpha_b = 0$ for all b. As seen above we have for all $b \in B(P, S, <, h) : b = \pi(\hat{b})$. Thus we have in $\operatorname{gr} A(n)$:

$$\sum_{b \in B_n} \alpha_b b = \sum_{b \in B_n} \alpha_b (\hat{b} + A_{n-1}) = \left(\sum_{b \in B_n} \alpha_b \hat{b} \right) + A_{n-1} = 0.$$

As B_n is linearly independent we obtain for all $b \in B_n \setminus B_{n-1} : \alpha_b = 0$. $\square$

Theorem 2.5.2. Assume that k is algebraically closed. Let H be a Hopf algebra generated by an abelian group G and skew primitive elements $a_1, \dots, a_t$ such that the subvector space of H spanned by $a_1, \dots, a_t$ is stable under the adjoint action of G. Then H has a PBW basis $(G, S, <, h)$.

Proof. First we may assume that for all $1 \leq i \leq t$

$$\Delta(a_i) = g_i \otimes a_i + a_i \otimes 1.$$

Let H_n be the subspace of H generated by all products of elements of G and at most n factors from $\{a_1, \dots, a_t\}$. This defines a Hopf algebra filtration of H. It is well known from [5, 34, 29] that we can decompose the associated graded Hopf algebra

$$\operatorname{gr} H \simeq R \# kG,$$

as graded Hopf algebras, where R is a braided graded Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ generated by the finite-dimensional Yetter-Drinfeld module $R(1) \subset P(R)$. As a G-module $R(1)$ is isomorphic to $ka_1 + \ldots + ka_t$ with the adjoint G action. Example 1.3.9 shows that the braiding on $R(1)$ is triangular because the group G is abelian. So by the PBW Theorem 2.2.4 we find a PBW basis $(\{1\}, S, <, h)$ of R. This implies that $(1\#G, S\#1, <, h)$ is a PBW basis of $\operatorname{gr} H$ and thus we find a PBW basis of H using the proposition above. □

2.6 Application to Nichols algebras of $U_q(\mathfrak{sl}_2)$-modules

As a second application of the PBW theorem we will deal with some interesting examples mentioned in [1] that are not of diagonal type by Remark 1.3.10, namely the Nichols algebras of simple $U_q(\mathfrak{sl}_2)$ modules of low dimension (and type $+1$). We will need the following lemma on Lyndon words that contain only two different letters.

Lemma 2.6.1. Let $X = \{x_0, x_1\}, x_0 < x_1$ and assume that $u \in \mathbb{X} \setminus X$ is a Lyndon word. Then there exist natural numbers $r \in \mathbb{N}, l_1, \dots, l_r, m_1, \dots, m_r \geq 1$ such that

$$u = x_0^{l_1} x_1^{m_1} \ldots x_0^{l_r} x_1^{m_r}.$$

For all $1 \leq i \leq r$ we have $l_i \leq l_1$ and if $l_i = l_1$ then also $m_i \geq m_1$.

Proof. It is obvious that the given decomposition of u exists and that the l_i, m_i are uniquely determined by u. For every $1 < i \leq r$ we have

$$x_0^{l_1} x_1^{m_1} \ldots x_0^{l_r} x_1^{m_r} = u < x_0^{l_i} x_1^{m_i} \ldots x_0^{l_r} x_1^{m_r}.$$

This implies that $l_i \leq l_1$. If we have $l_i = l_1$ we can cancel the x_0 on the left side and obtain

$$x_1^{m_1} x_0^{l_2} x_1^{m_2} \ldots x_0^{l_r} x_1^{m_r} < x_1^{m_i} x_0^{l_{i+1}} x_1^{m_{i+1}} \ldots x_0^{l_r} x_1^{m_r}.$$

From this we get $m_1 \leq m_i$. □

Assume that $\operatorname{char} k = 0$. Let $q \in k$ be not a root of unity and $(M, c) = L(n, +1)$ be the simple $U_q(\mathfrak{sl}_2)$ module of dimension $n+1$ and type $+1$ with braiding c induced by the quasi-$\mathcal{R}$-matrix and a function f as in Example 1.3.7. Denote its natural basis (see e.g. [15]) by $x_0, \dots, x_n$ and order this basis by $x_0 < \dots < x_n$. Then the braiding is left (and right) triangular with respect to this basis $X = \{x_0, \dots, x_n\}$. To compute the relations in low degrees from the matrix of the braiding we used Maple.

1. $n = 1, f(\frac{\alpha}{2}, \frac{\alpha}{2}) = q^{-2}$: As for example shown in [5], $\mathcal{B}(M, c)$ is a quadratic algebra. The relation in degree two is

   $$x_0x_1 - qx_1x_0 \quad = 0.$$

   Thus the set of PBW generators S contains only Lyndon words in x_0, x_1 that do not have x_0x_1 as a subword. Using Lemma 2.6.1 we see that this implies $S = \{x_0, x_1\}$. As the diagonal coefficients of c are powers of q by Lemma 2.2.6 all elements have infinite height. Thus the elements of the form $x_1^i x_0^j, i, j \in \mathbb{N}_0$ form a basis for $\mathcal{B}(M, c)$.

2. $n = 1, f(\frac{\alpha}{2}, \frac{\alpha}{2}) = q^{-1}$: In this case there are no relations in degree two and the relations in degree 3 are

   $$\begin{aligned} x_0x_1^2 - (q+1)x_1x_0x_1 + qx_1^2x_0 &= 0, \\ x_0^2x_1 - (q+1)x_0x_1x_0 + qx_1x_0^2 &= 0. \end{aligned}$$

   The set S contains all Lyndon words of x_0, x_1 that do not have $x_0^2x_1$ or $x_0x_1^2$ as a subword. Assume we have such a Lyndon word $u \in \mathbb{X} \setminus X$. Then each "block" $x_0^{l_i}x_1^{m_i}$ from Lemma 2.6.1 has to be of the form x_0x_1. This means $u = (x_0x_1)^r$ and because u is Lyndon we obtain $u = x_0x_1$.

   This leaves $S \subset \{x_0, x_0x_1, x_1\}$. As these words cannot be expressed as a linear combination of bigger words we have $S = \{x_0, x_0x_1, x_1\}$ and again every element of S has infinite height. In particular all elements of the form $x_1^i(x_0x_1)^jx_0^k, i, j, k \in \mathbb{N}_0$ form a basis of $\mathcal{B}(M, c)$ and the defining relations are exactly those listed above.

3. $n = 2, f(0, 0) = q^{-2}$: Here we have the following relations in degree two:

   $$\begin{aligned} x_0x_1 - q^2x_1x_0 &= 0, \\ x_1x_2 - q^2x_2x_1 &= 0, \\ x_0x_2 + (q^2-1)x_1x_1 - x_2x_0 &= 0. \end{aligned}$$

So all words in S are Lyndon words in x_0, x_1, x_2 that do not contain one of x_0x_1, x_1x_2, x_0x_2 as a subword. This means $S = \{x_0, x_1, x_2\}$. Again by Lemma 2.2.6 all elements of S have infinite height and thus the elements of the form $x_2^i x_1^j x_0^k, i, j, k \in \mathbb{N}_0$ form a basis of $\mathcal{B}(M, c)$. We see also that it is a quadratic algebra.

4. $n = 1, f(\frac{\alpha}{2}, \frac{\alpha}{2}) = v^{-2}$, where $v^3 = q$: Here we have no relations in degree 2 and 3. The relations in degree 4 are

$$\begin{aligned}
x_0x_1^3 - \frac{v^4+v^2+1}{v}x_1x_0x_1^2 + (v^4+v^2+1)x_1^2x_0x_1 - v^3x_1^3x_0 &= 0,\\
x_0^2x_1^2 - v\frac{v^4+v^2+1}{1+v^2}x_0x_1x_0x_1 - \frac{v^6-1}{v^2(v^2+1)}x_0x_1^2x_0 - &\\
\frac{v^6-1}{v^2(v^2+1)}x_1x_0^2x_1 + \frac{2v^4+v^2+1}{v^3(1+v^2)}x_1x_0x_1x_0 - x_1^2x_0^2 &= 0,\\
x_0^3x_1 - \frac{v^4+v^2+1}{v}x_0^2x_1x_0 + (v^4+v^2+1)x_0x_1x_0^2 - v^3x_1x_0^3 &= 0.
\end{aligned}$$

By combining these relations we obtain two new relations with leading words $x_0x_1x_0x_1^2$ and $x_0^2x_1x_0x_1$ (the coefficients are not zero in both cases as v is not a root of unity). So S contains all Lyndon words in x_0, x_1 that do not contain $x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_0x_1x_0x_1^2$ and $x_0^2x_1x_0x_1$. We show now that this implies $S \subset \{x_0, x_0^2x_1, x_0x_1, x_0x_1^2, x_1\}$:
Assume that we have such a Lyndon word $u \in S \setminus X$. Write $u = a_1 \dots a_r$ with $a_i = x_0^{l_i}x_1^{m_i}$ as in Lemma 2.6.1. Of course we have $a_i \in \{x_0^2x_1, x_0x_1, x_0x_1^2\}$ for all $0 \leq i \leq r$ and not all of the a_i are equal. We want to show that $r = 1$, so assume $r > 1$. First consider the case $a_1 = \dots = a_s = x_0^2x_1$ and $a_{s+1} \neq x_0^2x_1$. If $l_{s+1} = l_1$ then we have $m_{s+1} > m_1 = 1$. This means that a_{s+1} has the subword $x_0^2x_1^2$, which is not possible. If $l_{s+1} \neq l_1$ we have $l_{s+1} < l_1$ and thus a_{s+1} begins with x_0x_1. Then a_sa_{s+1} and hence also u have the subword $x_0^2x_1x_0x_1$ - a contradiction. As a second case assume $a_1 = \dots = a_s = x_0x_1$ and $a_{s+1} \neq x_0x_1$. Then a_{s+1} begins with $x_0x_1^2$ and thus a_sa_{s+1} has the subword $x_0x_1x_0x_1^2$ - a contradiction. Finally consider the case $a_1 = \dots = a_s = x_0x_1^2$ and $a_{s+1} \neq x_0x_1^2$. Then a_{s+1} begins with $x_0x_1^3$ - again a contradiction. This finishes the proof.

Now as all the remaining words have degrees ≤ 3 we see that actually all of them are contained in S. So $\mathcal{B}(M, c)$ has a basis made up of all elements of the form $x_1^i(x_0x_1^2)^j(x_0x_1)^l(x_0^2x_1)^m x_0^n, i, j, l, m, n \in \mathbb{N}_0$. Furthermore the defining relations are exactly those listed above.

5. $n = 3, f(\frac{\alpha}{2}, \frac{\alpha}{2}) = q^{-2}$: In this case the space of relations of degree two is generated by the elements

$$\begin{aligned} x_0x_1 - q^3x_1x_0 &= 0, \\ x_0x_2 + \frac{1-q^4}{q}x_1^2 - x_2x_0 &= 0, \\ q^3x_0x_3 + q^2(q^2+1-q^4)x_1x_2 + (q-q^3-q^5)x_2x_1 - x_3x_0 &= 0, \\ x_1x_3 + \frac{1-q^6}{q(q^2+1)}x_2^2 - x_3x_1 &= 0, \\ x_2x_3 - q^3x_3x_2 &= 0. \end{aligned}$$

By combining these relations one obtains the additional relations

$$\begin{aligned} (q^4-q^2+1)x_1x_2x_2 - q(q^6+1)x_2x_1x_2 + (q^4-q^2+1)q^4x_2x_2x_1 &= 0, \\ x_1x_1x_2 - q(q^2+1)x_1x_2x_1 + q^4x_2x_1x_1 &= 0. \end{aligned}$$

As q is a root of unity, the leading coefficients in these relations are not zero: the zeros of $X^4 - X^2 + 1$ are primitive 12-th roots of unity as

$$X^{12} - 1 = (X^4 - X^2 + 1)(X^2 + 1)(1 - X^6).$$

Thus S can only contain Lyndon words in x_0, x_1, x_2, x_3 that do not contain a subword from the following list:

$$x_0x_1, x_0x_2, x_0x_3, x_1x_3, x_2x_3, x_1^2x_2, x_1x_2^2.$$

These are exactly x_0, x_3 and all Lyndon words in x_1 and x_2 that do not contain $x_1^2x_2$ and $x_1x_2^2$. It follows that $S \subset \{x_0, x_1, x_2, x_3, x_1x_2\}$. None of these words can be expressed by standard-bigger ones as we can see from the relations of degree 2. Thus $S = \{x_0, x_1, x_2, x_3, x_1x_2\}$ and the elements of the form $x_3^a x_2^b (x_1x_2)^c x_1^d x_0^e, a, b, c, d, e \in \mathbb{N}_0$ form a basis of $\mathcal{B}(M, c)$. Furthermore $\mathcal{B}(M, c)$ is a quadratic algebra.

Note that in every example the Nichols algebra has finite Gelfand-Kirillov dimension. So these simple $U_q(\mathfrak{sl}_2)$-modules can also be found in Table 4.1 (page 101). Actually these are all cases of simple $U_q(\mathfrak{sl}_2)$-modules of type $+1$ (and functions f of exponential type) that have a Nichols algebras of finite Gelfand-Kirillov dimension.

Chapter 3

A characterization of triangular braidings

In Definition 1.3.5 triangular braidings are characterized by a combinatorial property. Already in Chapter 2 a close connection to diagonal braidings and pointed Hopf algebras with abelian coradical became apparent. In this chapter (Theorem 3.3.6) a further aspect of this connection is established. We show that triangular braidings are exactly those braidings that arise from certain Yetter-Drinfeld module structures over pointed Hopf algebras with abelian coradical.
This offers a better understanding of the mathematical context of triangular braidings. We will give explicit constructions in the case of $U_q(\mathfrak{g})$-modules. This will open a new way to study Nichols algebras in Chapter 4.

3.1 The reduced FRT Hopf algebra

An important tool in this chapter will be the reduced FRT Hopf algebra of a rigid braiding. In the case of the FRT bialgebra a similar construction was given in [33]. The construction given here has the advantage that it generalizes easily to the FRT Hopf algebra in the rigid case. First we will recall some facts on coquasitriangular bialgebras.

Definition 3.1.1. A coquasitriangular bialgebra $(H, \nabla, \eta, \Delta, \varepsilon, r)$ is a bialgebra together with a convolution invertible bilinear form $r \in (H \otimes H)^*$ satisfying

$$\nabla\tau =: \nabla^{op} = r \star \nabla \star r^{-1}, \text{ and}$$

$$r \circ (\nabla \otimes \mathrm{id}_H) = r_{13} \star r_{23}, \text{ and } r \circ (\mathrm{id}_H \otimes \nabla) = r_{13} \star r_{12},$$

where we define $r_{12}, r_{23}, r_{13} \in (H \otimes H \otimes H)^*$ by

$$r_{12} := r \otimes \varepsilon, \;\; r_{23} := \varepsilon \otimes r, \;\text{ and }\; r_{13}(g \otimes h \otimes l) := \varepsilon(h) r(g \otimes l)$$

for all $g, h, l \in H$.

Remark 3.1.2. Let $(H, \nabla, \eta, \Delta, \varepsilon, r)$ be a coquasitriangular Hopf algebra with antipode $\mathbf{S}$.

1. $\mathbf{S}^2$ is a coinner automorphism of H. In particular $\mathbf{S}$ is invertible.
2. $r \circ (\eta \otimes \mathrm{id}_H) = \varepsilon$ and $r \circ (\mathrm{id}_H \otimes \eta) = \varepsilon$.
3. $r \circ (\mathbf{S} \otimes \mathrm{id}_H) = r^{-1}$, $\; r^{-1} \circ (\mathrm{id}_H \otimes \mathbf{S}) = r$, $\; r \circ (\mathbf{S} \otimes \mathbf{S}) = r$.

If H is a coquasitriangular bialgebra, the second and third axiom from Definition 3.1.1 read for $a, b, c \in H$:

$$r(ab \otimes c) = r(a \otimes c_{(1)}) r(b \otimes c_{(2)}), \;\; r(a \otimes bc) = r(a_{(2)} \otimes b) r(a_{(1)} \otimes c)$$

Remark 3.1.3. Assume that H is a coquasitriangular bialgebra and that M is a H-comodule. It is a well-known fact that then M becomes a Yetter-Drinfeld module over H with action given by

$$h \cdot m := r(m_{(-1)} \otimes h) m_{(0)}$$

for all $h \in H, m \in M$.

Next we will introduce reduced versions H^{red} of coquasitriangular bialgebras H such that comodules over H still are Yetter-Drinfeld modules over H^{red}.

Definition 3.1.4. Let H be a coquasitriangular bialgebra. Define the right radical of H as

$$J_H := \{h \in H | \forall g \in H : r(g \otimes h) = 0\}.$$

Lemma 3.1.5. Let H be a coquasitriangular bialgebra.

1. The right radical is a biideal in H.
2. If H is a Hopf algebra, then the right radical is stable under $\mathbf{S}$ and $\mathbf{S}^{-1}$.

Proof. The properties of r imply that the map

$$H \to (H^\circ)^{cop}, h \mapsto r(- \otimes h)$$

is a (well-defined) morphism of bialgebras (resp. Hopf algebras). J_H is the kernel of this map. $\square$

The following easy lemma provides a useful characterization of the right radical.

Lemma 3.1.6. Let H be a coquasitriangular bialgebra generated as an algebra by a subset $X \subset H$. Then for every coideal $J \subset H$ we have $r(H, J) = 0$ if and only if $r(X, J) = 0$ and thus

$$J_H = \sum \{J | J \subset H \text{ is a coideal with } r(X, J) = 0\}.$$

Now we define the reduced version of a coquasitriangular bialgebra (Hopf algebra).

Definition 3.1.7. Let H be a coquasitriangular bialgebra (Hopf algebra) and J_H its right radical. Define

$$H^{red} := H/J_H,$$

the factor bialgebra (Hopf algebra).

Note that if H is a coquasitriangular Hopf algebra, then H and H^{red} have bijective antipodes. In order to define the reduced FRT constructions and prove their universal properties we will now recall necessary the facts on the usual FRT constructions. The FRT construction was first considered by Faddeev, Reshetikhin, and Takhtadzhyan in [9]. For a detailed description see [19, VIII.6.].

Theorem 3.1.8. Let (M, c) be a finite dimensional braided vector space.

- There is a coquasitriangular bialgebra $A(c)$ - called the FRT bialgebra of c - such that M is a left $A(c)$-comodule and the braiding c equals the braiding on M induced by the coquasitriangular structure of $A(c)$.
- For all bialgebras B having M as a Yetter-Drinfeld module such that the induced braiding equals c there is a unique morphism of bialgebras $\phi : A(c) \to B$ such that

$$\delta_B = (\phi \otimes \mathrm{id}_M)\delta_{A(c)} \text{ and } \forall u \in A(c), m \in M : u \cdot m = \phi(u) \cdot m.$$

The algebra $A(c)$ is generated by the smallest subcoalgebra $C \subset A(c)$ satisfying $\delta_{A(c)}(M) \subset C \otimes M$.

An important question is, under which assumptions we can define an Hopf algebra analogue of the FRT bialgebra. The necessary condition is that the braiding is rigid.

Definition 3.1.9. For any finite dimensional vector space M define maps

$$\mathrm{ev} : M^* \otimes M \to k,\ \mathrm{ev}(\phi \otimes m) := \phi(m),$$

$$\mathrm{db} : k \to M \otimes M^*,\ \mathrm{db}(1) := \sum_{i=1}^{n} m_i \otimes m^i,$$

where $m_1, \dots, m_n$ form a basis of M and $m^1, \dots, m^n$ is the dual basis of M^*.

Definition 3.1.10. A braided vector space (M, c) will be called *rigid* if it is finite dimensional and the map $c^\flat$ defined by

$$M^* \otimes M \xrightarrow{M^* \otimes M \otimes \mathrm{db}} M^* \otimes M \otimes M \otimes M^* \xrightarrow{M^* \otimes c \otimes M^*} M^* \otimes M \otimes M \otimes M^* \xrightarrow{ev \otimes M \otimes M^*} M \otimes M^*$$

is an isomorphism.

Now we can formulate the Hopf algebra version of Theorem 3.1.8. It was proved for symmetries by Lyubashenko [28]. A version for general (rigid) braidings can be found in [39].

Theorem 3.1.11. Let (M, c) be a rigid braided vector space.

- There is a coquasitriangular Hopf algebra $H(c)$ - the FRT Hopf algebra of c - such that M is a left $H(c)$-comodule and the braiding c equals the braiding on M induced by the coquasitriangular structure of $H(c)$.
- For all Hopf algebras H having M as a Yetter-Drinfeld module such that the induced braiding equals c there is a unique morphism of Hopf algebras $\psi : H(c) \to H$ such that

$$\delta_H = (\psi \otimes \mathrm{id}_M)\delta_{H(c)} \text{ and } \forall u \in H(c), m \in M : u \cdot m = \psi(u) \cdot m$$

Let $C \subset H(c)$ be the smallest subcoalgebra satisfying $\delta_{H(c)}(M) \subset C \otimes M$. Then the algebra $H(c)$ is generated by $C + \mathbf{S}(C)$.

Proof. The existence of the coquasitriangular Hopf algebra $H(c)$ is proved in [39, Theorem 3.2.9]. The universal property we give is a bit stronger than the one given there. Let H be a Hopf algebra as in the second part of the theorem. Fix a basis $(x_i)_{i \in I}$ of M and elements $(T_i^j)_{i,j \in I}$ of H such that

$$\delta(x_i) = \sum_{j \in I} T_i^j \otimes x_j,\ \ \Delta(T_i^j) = \sum_{l \in I} T_i^l \otimes T_l^j.$$

Furthermore fix scalars $(B_{ij}^{kl})_{i,j,k,l \in I}$ such that

$$c(x_i \otimes x_j) = \sum_{k,l \in I} B_{ij}^{kl} x_k \otimes x_l.$$

By [39, Lemma 3.2.11] it suffices to show that for all $i, j, k, l \in I$ the relation

$$\sum_{m,n\in I} T_n^l T_m^j B_{ik}^{nm} = \sum_{m,n\in I} B_{nm}^{lj} T_i^m T_k^n$$

holds. As the induced Yetter-Drinfeld braiding is c we see that the action is given for all $i, j, k \in I$ by

$$T_i^j x_l = \sum_{k\in I} B_{il}^{kj} x_k.$$

Using this the relations between the T_i^j follow from the Yetter-Drinfeld condition

$$(T^j_{i\,(1)} x_k)_{(-1)} T^j_{i\,(2)} \otimes (T^j_{i\,(1)} x_k)_{(0)} = T^j_{i\,(1)} x_{k(-1)} \otimes T^j_{i\,(2)} x_{k(0)}.$$

□

For a braided vector space (M, c) we define the reduced FRT bialgebra by

$$A^{red}(c) := (A(c))^{red}$$

and if (M, c) is rigid define the reduced FRT Hopf algebra by

$$H^{red}(c) := (H(c))^{red}.$$

Definition 3.1.12. Let H be a bialgebra and $M_1, \dots, M_s$ H-modules. We will call H $M_1, \dots, M_s$*-reduced* if (0) is the only coideal of H annihilating all the M_i.

The reduced FRT constructions are characterized by universal properties:

Theorem 3.1.13. Let (M, c) be a finite dimensional braided vector space.

1. M is a Yetter-Drinfeld module over $A^{red}(c)$ such that the induced braiding is c. $A^{red}(c)$ is M-reduced.

2. For every bialgebra A having M as a Yetter-Drinfeld module such that the induced braiding is c and such that A is M-reduced there is a unique monomorphism of bialgebras $\phi : A^{red}(c) \to A$ such that

$$\delta_A = (\phi \otimes M)\delta_{A^{red}(c)} \text{ and } \forall u \in A^{red}(c), m \in M : u \cdot m = \phi(u) \cdot m.$$

3. Assume (M, c) is rigid. M is a Yetter-Drinfeld module over $H^{red}(c)$ such that the induced braiding is c. $H^{red}(c)$ is M, M^*-reduced.

4. Assume (M, c) is rigid. For every Hopf algebra H having M as a Yetter-Drinfeld module such that the induced braiding is c and such that H is M, M^*-reduced there is a unique monomorphism of Hopf algebras $\psi : H^{red}(c) \to H$ such that

$$\delta_H = (\psi \otimes M)\delta_{H^{red}(c)} \text{ and } \forall u \in H^{red}(c), m \in M : u \cdot m = \psi(u) \cdot m.$$

Proof. Parts one and three are trivial. We will deal with parts two and four simultaneously: Using the universal property of the FRT constructions we find morphisms of bialgebras

$$\hat{\phi} : A(c) \to A \text{ and } \hat{\psi} : H(c) \to H.$$

which are compatible with the action and the coaction. The right radical of $A(c)$ (resp. $H(c)$) is the maximal coideal annihilating M (resp. M and M^*). Thus the image of the right radical under $\hat{\phi}$ (resp. $\hat{\psi}$) is again a coideal annihilating M (resp. M, M^*). As A (resp. H) is M (resp. M, M^*)-reduced we see that the right radical is mapped to (0). This means that $\hat{\phi}$ (resp. $\hat{\psi}$) factorize over the reduced FRT constructions. These induced maps are compatible with action and coaction. Injectivity of the induced maps follows because of the maximality of the right radical mentioned above. $\square$

Remark 3.1.14. In [33] Radford defines a reduced FRT bialgebra $A^{red}(R)$ for Yang-Baxter operators R on finite dimensional vector spaces M, that is automorphisms R of $M \otimes M$ satisfying the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

It is well known that R satisfies the quantum Yang-Baxter equation if and only if $c := R\tau$ is a braiding (satisfies the braid equation). It is easy to see from the universal properties of $A^{red}(c)$ and $A^{red}(R)$ that if $c = R\tau$ we have

$$A^{red}(c) \simeq A^{red}(R)^{cop}.$$

Remark 3.1.15. Suppose that M is a Yetter-Drinfeld module over a Hopf algebra H with bijective antipode and denote the braiding on M by c. It is easily seen that then $H^{red}(c)$ is a sub-quotient (i.e. a Hopf algebra quotient of a Hopf subalgebra) of H.

Example 3.1.16. *$H^{red}(c)$ for braidings of group type.* Let G be a group and M a finite dimensional Yetter-Drinfeld module of G. Define

$$C := \{g \in G | \exists m \in M : \delta(m) = g \otimes m\}$$

and let H be the subgroup of G generated by C. Moreover set

$$N := \{g \in H | \forall m \in M : gm = m\}.$$

Obviously M becomes a Yetter-Drinfeld module over the sub-quotient H/N. It is easy to show that $k(H/N)$ is the reduced FRT construction.

3.2 When is $H^{red}(c)$ pointed?

As a first step to our characterization of triangular braidings we will answer the question from the heading of this section. In the case of the FRT bialgebra the answer to this question was given by Radford.

Theorem 3.2.1 ([33], Theorem 3). Let (M, c) be a finite dimensional braided vector space. The following are equivalent:

1. $A^{red}(c)$ is pointed.
2. There is a flag of subspaces $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ such that for all $1 \leq i \leq r$ $\dim M_i = i$ and

$$c(M_i \otimes M) \subset M \otimes M_i.$$

3. There is a flag of $A^{red}(c)$ left subcomodules $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ such that for all $1 \leq i \leq r$ $\dim M_i = i$.

We will show now that if (M, c) is rigid we have a similar statement for the reduced FRT Hopf algebra $H^{red}(c)$.

Lemma 3.2.2. Let (M, c) be a rigid braided vector space and $N \subset M$ a subspace such that

$$c(N \otimes M) \subset M \otimes N.$$

Then we have

$$c^\flat(M^* \otimes N) \subset N \otimes M^*.$$

Proof. Choose a complement X of N. Then

$$\operatorname{Im} \mathrm{db} \subset N \otimes X^\perp \oplus X \otimes N^\perp.$$

Using the definition of $c^\flat$ in 3.1.10 it is easy to see that this implies the lemma (here $N^\perp := \{\varphi \in M^* | \varphi(N) = 0\}$ for $N \subset M$). □

We will need the following well known statement:

Proposition 3.2.3. Let H be a coquasitriangular Hopf algebra and M a left H-comodule. Then M^* with the coaction defined by the equation

$$\forall \varphi \in M^*, m \in M : \varphi_{(-1)}\varphi_{(0)}(m) = \mathbf{S}^{-1}(m_{(-1)})\varphi(m_{(0)})$$

together with the maps ev, db forms a left dual of M in the categorical sense (see e.g. [19, XIV.2.1]). In particular $c^\flat$ is the inverse of the braiding c_{M,M^*} between M and M^*.

Proof. An easy calculation shows that ev, db are indeed colinear. Thus they define a duality. The proof that $c^\flat$ is indeed the inverse of c_{M,M^*} is analogous to that of [19, XIV.3.1]. □

The following lemma is already used in [33]. We include a proof for completeness.

Lemma 3.2.4. Let H be a bialgebra generated (as an algebra) by a subcoalgebra $C \subset H$. If C is pointed, then so is H.
In this case the coradical of H is generated by the coradical of C as an algebra.

Proof. Let $(C_n)_{n\geq 0}$ be the coradical filtration of C. Denote by D_0 the subalgebra of H generated by C_0. As C is pointed we have $D_0 \subset kG(H)$. Consider the subsets

$$D_n := \wedge^n D_0 \;\; \forall n \geq 0.$$

As D_0 is a subbialgebra of H the D_n define a filtration of the bialgebra

$$D := \cup_{n\geq 0} D_n.$$

Now because $C_0 \subset D_0$ we have $C_n \subset D_n$ for all $n \geq 0$ and then $C \subset D$. This means $D = H$ and the D_n define a bialgebra filtration of H. We find

$$kG(H) \subset \operatorname{Corad} H \subset D_0 \subset kG(H)$$

and see that H is pointed. □

Theorem 3.2.5. Let (M, c) be a rigid braided vector space. The following are equivalent:

1. $H^{red}(c)$ is pointed.

2. There is a pointed Hopf algebra H having M as a Yetter-Drinfeld module such that the induced braiding is c.

3. There is a flag of left $H^{red}(c)$ subcomodules $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ such that for all $1 \leq i \leq n$ $\dim M_i = i$.

4. There is a flag of subvector spaces $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ such that for all $1 \leq i \leq n$ $\dim M_i = i$ and
$$c(M_i \otimes M) \subset M \otimes M_i.$$

Proof. It is clear that the first item implies the second. If H is as in (2) (e.g. $H = H^{red}(c)$) we find a series of subcomodules
$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$
with $\dim M_i = i$ for all $1 \leq i \leq r$. In view of the definition of the braiding for Yetter-Drinfeld modules this means that (1) implies (3) (and hence also (4)) and that (2) implies (4). We still have to show that (4) implies (1).
So now assume that (4) holds. For all $1 \leq i \leq r$ choose $m_{r+1-i} \in M_i \setminus M_{i-1}$ arbitrarily (thus $m_i, \ldots, m_r \in M_i$). This defines a basis of M and let $m^1, \ldots, m^r$ be the dual basis. We find elements $t_{ij} \in H(c), 1 \leq i, j \leq r$ satisfying
$$\Delta(t_{ij}) = \sum_{l=1}^{r} t_{il} \otimes t_{lj} \text{ and } \delta(m_i) = \sum_{l=1}^{r} t_{il} \otimes m_l.$$
Using the definition of the coaction of M^* we see that
$$\delta(m^i) = \sum_{l=1}^{r} \mathbf{S}^{-1}(t_{li}) \otimes m^l.$$
Now define
$$J := \text{k-span}\{t_{ij} | 1 \leq j < i \leq r\}.$$
J is a coideal of $H(c)$ and we will show $J \subset J_{H(c)}$. For all $1 \leq i, k \leq r$ we have
$$c(m_k \otimes m_i) \in c(M_k \otimes M) \subset M \otimes M_k$$
and on the other hand
$$c(m_k \otimes m_i) = \sum_{l=1}^{r} t_{kl} m_i \otimes m_l = \sum_{j,l=1}^{r} r(t_{ij}, t_{kl}) m_j \otimes m_l$$
This implies $r(t_{ij}, t_{kl}) = 0$ for $l < k$. Moreover because $c(M_i \otimes M) \subset M \otimes M_i$ we have by Lemma 3.2.2 that $c^\flat(M^* \otimes M_i) \subset M_i \otimes M^*$ and thus by Proposition 3.2.3 that
$$c_{M,M^*}(M_i \otimes M^*) \subset M^* \otimes M_i$$

In the same manner as above we obtain $r(\mathbf{S}^{-1}(t_{ij}), t_{kl}) = 0$ for $l < k$. Now by Theorem 3.1.11 the algebra $H(c)$ is generated by the t_{ij} and the $\mathbf{S}(t_{ij})$. Thus it is also generated by the $\mathbf{S}^{-1}(t_{ij})$ and the t_{ij}. Lemma 3.1.6 allows us to conclude that $r(H, J) = 0$ and thus $J \subset J_{H(c)}$. As $J_{H(c)}$ is stable under $\mathbf{S}$ we obtain $J + \mathbf{S}(J) \subset J_{H(c)}$.
To see that $H^{red}(c)$ is pointed it suffices to show that the coalgebra C spanned by the images of $\overline{t_{ij}}, \mathbf{S}(\overline{t_{kl}})$ of $t_{ij}, \mathbf{S}(t_{kl})$ in $H^{red}(c)$ is pointed (Lemma 3.2.4). For this define subsets $C_n, n \geq 0$ by

$$C_n := \text{k-span}\{\overline{t_{ij}}, \mathbf{S}(\overline{t_{ij}}) | 1 \leq i \leq j \leq i + n \leq r\} \subset H^{red}(c).$$

Since $J + \mathbf{S}(J) \subset J_{H(c)}$, we find that the C_n define a coalgebra filtration of C. Thus

$$\operatorname{Corad} C \subset C_0 = \text{k-span}\{\overline{t_{ii}}, \mathbf{S}(\overline{t_{ii}}) | 1 \leq i \leq r\}.$$

As $\Delta(\overline{t_{ii}}) = \overline{t_{ii}} \otimes \overline{t_{ii}}$ and $\Delta(\mathbf{S}(\overline{t_{ii}})) = \mathbf{S}(\overline{t_{ii}}) \otimes \mathbf{S}(\overline{t_{ii}})$ in $H^{red}(c)$ we find that C is pointed. $\square$

For future use we remark that the coradical of $H^{red}(c)$ is generated by the elements $\overline{t_{ii}}, \mathbf{S}(\overline{t_{ii}})$.

3.3 The reduced FRT construction for triangular braidings

In this section we will consider the reduced FRT constructions of (right) triangular braidings and obtain a characterization of triangular braidings. First we prove that triangular braidings are indeed rigid, i.e. the notion of a (reduced) FRT Hopf algebra makes sense. Note that c is a braiding if and only if $\tau c \tau$ is a braiding, where τ is the flip map.

Proposition 3.3.1. Let (M, c) be a finite dimensional braided vector space.

1. c is left triangular if and only if c^{-1} is right triangular.

2. c is left triangular if and only if $\tau c \tau$ is right triangular.

Proof. (2) is trivial. Thus for (1) it suffices to show the if-part. Assume c^{-1} is right triangular and adopt the notation from the definition. Define $M_{>x} := \text{k-span}\{z \in X | z>x\}$. We see from the definition that

$$c(y \otimes x) = \beta_{xy}^{-1} x \otimes y + c\left(\sum_{z>x} w_{x,y,z} \otimes z\right) \in \beta_{xy}^{-1} x \otimes y + c(M \otimes M_{>x}).$$

It is now easy to show by downward induction on x (along the order on X) that

$$c(y \otimes x) \in \beta_{xy}^{-1} x \otimes y + M_{>x} \otimes M.$$

Thus c is left triangular. □

Lemma 3.3.2. Let (M, c) be a (left or right) triangular braided vector space. Then (M, c) and (M, c^{-1}) are rigid.

Proof. In both cases it suffices to show that (M, c) is rigid. Assume (M, c) is left triangular with respect to the basis X. Let $(\varphi_x)_{x \in X}$ denote the dual basis ($\varphi_x(y) = \delta_{xy}$). Then

$$\begin{aligned} c^\flat \tau(x \otimes \varphi_y) &= c^\flat(\varphi_y \otimes x) \\ &= \sum_{z \in X} (\varphi_y \otimes M) c(x \otimes z) \otimes \varphi_z \\ &= \gamma_{x,y} x \otimes \varphi_y + \sum_{z \in X, z' > z} \varphi_y(z') v_{x,z,z'} \otimes \varphi_z \\ &\in \gamma_{x,y} x \otimes \varphi_y + \sum_{z<y} v_{x,z,y} \otimes \varphi_z. \end{aligned}$$

This means that the map $c^\flat \tau$ has upper triangular representing matrix with respect to the basis

$$x_1 \otimes \varphi_{x_1}, \ldots, x_r \otimes \varphi_{x_1}, x_1 \otimes \varphi_{x_2}, \ldots, x_r \otimes \varphi_{x_2}, \ldots, x_1 \otimes \varphi_{x_r}, \ldots, x_r \otimes \varphi_{x_r}$$

(where we assumed that the elements of X are $x_1 < x_2 < \ldots < x_r$). The diagonal entries are $\gamma_{x,y} \neq 0$ and thus the matrix is invertible. This shows that $c^\flat$ is an isomorphism. A similar proof works for right triangular braidings. □

These results together with the Theorems 3.2.1 and 3.2.5 already show that the reduced FRT construction $A^{red}(c)$ (resp. $H^{red}(c)$) is pointed if c is right triangular. We will refine this knowledge now by describing the flag of comodules we used in the proof of 3.2.5 more exactly.

Definition 3.3.3. Let G be an abelian monoid and M a G-module. We say G *acts diagonally on* M if M is the direct sum of simultaneous eigenspaces under the action of G, this means:

$$M = \bigoplus_{\chi \in \hat{G}} \{m \in M | \forall g \in G : gm = \chi(g)m\}.$$

Proposition 3.3.4. Let H be a pointed bialgebra with abelian coradical such that for all $g \in G := G(H)$ the map $H \to H, h \mapsto hg$ is injective. Let $M \in {}^H_H\mathcal{YD}$ be such that G acts diagonally on M. Then there is a series of H subcomodules and G submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that $\dim M_i = i$ for all $1 \leq i \leq r$.

Proof. Consider modules $N \in {}^H\mathcal{M} \cap {}_G\mathcal{M}$ that have an eigenspace decomposition as in the lemma and satisfy the following compatibility condition:

$$(gn)_{(-1)} g \otimes (gn)_{(0)} = gn_{(-1)} \otimes gn_{(0)} \ \forall g \in G, n \in N.$$

It suffices to show that every such module N contains a one dimensional H subcomodule that is also a G submodule (note that the objects considered in the lemma are of this type).
So pick a simple subcomodule of N. As H is pointed this is spanned by $n_0 \in N$ and we find $g \in G$ such that $\delta(n_0) = g \otimes n_0$. Consider the subcomodule

$$0 \neq X := \{n \in N | \delta(n) = g \otimes n\} \subset N.$$

This is a G submodule as for $n \in X$ and $h \in G$ we have by the compatibility condition (and because G is abelian)

$$(hn)_{(-1)} h \otimes (hn)_{(0)} = hn_{(-1)} \otimes hn_{(0)} = hg \otimes hn = gh \otimes hn.$$

Now right multiplication with h is injective by assumption and we obtain $\delta(hn) = g \otimes hn$ showing that X is indeed a G submodule. A lemma from linear algebra tells us that because N is the direct sum of eigenspaces under the action of G, so is X. We find an element $n \in N$ that is an eigenvector under the action of G. Then kn is a one dimensional H subcomodule and G submodule. $\square$

Proposition 3.3.5. Let H, M be as in 3.3.4. Then we can find a basis $m_1, \ldots, m_r$ of M made up of eigenvectors under the action of $G(H)$ and elements $c_{ij} \in H, 1 \leq i \leq j \leq r$ such that

$$\delta(m_i) = \sum_{l=i}^{r} c_{il} \otimes m_l, \ \ \Delta(c_{ij}) = \sum_{l=i}^{j} c_{il} \otimes c_{lj}, \ \ \varepsilon(c_{ij}) = \delta_{ij}.$$

Proof. Take the series of comodules from 3.3.4. For all $1 \leq i \leq r$ we can choose an eigenvector $m_{r+1-i} \in M_i \setminus M_{i-1}$ (thus $M_i = \text{k-span}\{m_i, \dots, m_r\}$). Now we can find elements $c_{ij} \in H, 1 \leq i \leq j \leq r$ such that

$$\delta(m_i) = \sum_{l=i}^{r} c_{il} \otimes m_l.$$

The formulas for the comultiplication and the counit follow from the axioms of comodules. □

Theorem 3.3.6. Let (M, c) be a rigid braided vector space. The following are equivalent:

1. c is right triangular.
2. $H^{red}(c)$ is pointed with abelian coradical and $G(H^{red}(c))$ acts diagonally on M.
3. There is a pointed Hopf algebra H with abelian coradical having M as a Yetter-Drinfeld module such that the induced braiding is c and $G(H)$ acts diagonally on M.

Proof. Of course (2) implies (3). Assume c is right triangular with respect to the basis $m_1, \dots, m_r$ ordered by $m_1 < \dots < m_r$. If we define $M_i := \text{k-span}\{m_i, \dots, m_r\}$, then we have of course

$$c(M_i \otimes M) \subset M \otimes M_i \ \forall 1 \leq i \leq r.$$

Theorem 3.2.5 tells us that $H^{red}(c)$ is pointed. We adopt the notation from the proof of (4) $\Rightarrow$ (1) there. Then we obtain using the right triangularity of c:

$$\sum_{l=i}^{r} \overline{t_{il}} m_j \otimes m_l = c(m_i \otimes m_j) \in \alpha_{ij} m_j \otimes m_i + M \otimes \sum_{l=i+1}^{r} k m_l.$$

This means $\overline{t_{ii}} m_j = \alpha_{ij} m_j$ for all $1 \leq i, j \leq r$. As the $\overline{t_{ii}}$ and their inverses generate the coradical of $H^{red}(c)$ as an algebra we get that $G(H^{red}(c))$ acts diagonally on M.
We are left to show that the $G(H^{red}(c))$ is abelian. Let $g, h \in G(H^{red}(c))$, thus g and h act diagonally on M. Then $gh - hg$ acts as 0 on M, saying $k(gh - hg)$ is a coideal annihilating M. In the same way $g^{-1}h^{-1} - h^{-1}g^{-1}$ annihilates M and thus $k(gh - hg)$ annihilates M^*. As $H^{red}(c)$ is M, M^*-reduced, we get that $gh = hg$.

Now assume we are given a Hopf algebra as in (3). Let $m_1, \dots, m_r$ be the basis of M from Proposition 3.3.5 and also take the c_{ij} from there. Then we have

$$c(m_i \otimes m_j) \in c_{ii} m_j \otimes m_i + M \otimes \sum_{l>i} k m_l.$$

Now since $c_{ii} \in G(H)$ we obtain $c_{ii} m_j \in k m_j \setminus \{0\}$ and thus c is right triangular. □

3.4 Explicit constructions for $U_q(\mathfrak{g})$-modules

Assume for this section that the characteristic of the base field k is zero and that $q \in k$ is not a root of unity. Let $\mathfrak{g}$ be a complex finite-dimensional semi-simple Lie algebra with root system (V, Φ), weight lattice Λ and Π a basis of the root system. Furthermore fix a function $f : \Lambda \times \Lambda \to k^\times$ that satisfies Equation 1.1 in Example 1.3.7. For an integrable $U_q(\mathfrak{g})$-module M denote the braiding $c^f_{M,M}$ by c^f.
We will explicitly construct a Hopf algebra U and a Yetter-Drinfeld module structure over U on every integrable $U_q(\mathfrak{g})$-module such that the Yetter-Drinfeld braidings equal the $c^f_{M,N}$ defined by the quasi-$\mathcal{R}$-matrix and the function f. This is an important tool for our treatment of Nichols algebras of $U_q(\mathfrak{g})$-modules in Chapter 4. Furthermore it allows us to calculate the reduced FRT construction explicitly. A similar construction was mentioned in [38] for highest weight modules.
Since Lusztig [27] we know that $U_q^{\leq 0}(\mathfrak{g})$ decomposes as a Radford biproduct

$$U_q^{\leq 0}(\mathfrak{g}) = \mathcal{B}(V) \# k\Gamma.$$

Here $\Gamma \cong \mathbb{Z}\Phi$ is written multiplicatively identifying $\mu \in \mathbb{Z}\Phi$ with $K_\mu \in \Gamma$ as usual. $\mathcal{B}(V)$ is the Nichols algebra of the vector space $V := \oplus_{\alpha\in\Pi} k\hat{F}_\alpha$ with braiding

$$c(\hat{F}_\alpha \otimes \hat{F}_\beta) = q^{-(\alpha,\beta)} \hat{F}_\beta \otimes \hat{F}_\alpha.$$

The usual generators F_α as in Jantzens book are given by $\hat{F}_\alpha = K_\alpha F_\alpha$. The following easy lemma allows to define representations of the biproduct algebra.

Lemma 3.4.1. Let H be a Hopf algebra with bijective antipode and R a Hopf algebra in ${}^H_H\mathcal{YD}$. Let A be any algebra. The following data are equivalent:

- an algebra morphism $\psi : R\#H \to A$

- algebra morphisms $\rho : H \to A$ and $\varphi : R \to A$ such that:

  $$\forall h \in H, r \in R : \rho(h)\varphi(r) = \varphi(h_{(1)} \cdot r)\rho(h_{(2)}),$$

  where r resp. h run through a set of algebra generators of R resp. H.

In this case $\psi = \nabla_A(\varphi\#\rho)$ and $\varphi = \psi|R\#1, \rho = \psi|1\#H$.

Proof. The proof is straightforward and will be omitted. □

Now the Hopf algebra U and the Yetter-Drinfeld module structure will be constructed in 6 steps.

Step 1: Enlarge the Group.
As $\mathbb{Z}\Phi \subset \Lambda$ are free abelian groups of the same rank $|\Pi|$, the quotient $\Lambda/\mathbb{Z}\Phi$ is a finite group. Choose a set $X \subset \Lambda$ of representatives of the cosets of $\mathbb{Z}\Phi$. Define

$$G := \Gamma \times H,$$

where H denotes the free abelian group generated by the set X (written multiplicatively). For every $\lambda \in \Lambda$ there are unique elements $\alpha_\lambda \in \mathbb{Z}\Phi$ and $x_\lambda \in X$ such that

$$\lambda = \alpha_\lambda + x_\lambda.$$

Define for any $\lambda \in \Lambda$

$$L_\lambda := (K_{\alpha_\lambda}^{-1}, x_\lambda) \in G.$$

Note that for $\mu \in \mathbb{Z}\Phi, \lambda \in \Lambda$

$$L_{\lambda-\mu} = L_\lambda K_\mu.$$

Step 2: Define U
Now define a kG-coaction on V by setting

$$\delta_V(\hat{F}_\alpha) := K_\alpha \otimes \hat{F}_\alpha \text{ for all } \alpha \in \Pi.$$

Consider the action defined by

$$K_\alpha \hat{F}_\beta := q^{-(\beta,\alpha)}\hat{F}_\beta \text{ and } L_x\hat{F}_\beta := q^{(\beta,x)}\hat{F}_\beta$$

for all $\alpha, \beta \in \Pi, x \in X$. Obviously this defines a ${}^G_G\mathcal{YD}$ structure on V inducing the original braiding. The desired Hopf algebra is

$$U := \mathcal{B}(V)\#kG.$$

Step 3: The action of U on $U_q(\mathfrak{g})$-modules
Let M be an integrable $U_q(\mathfrak{g})$-module. Define the action of G on $m \in M_\lambda$ by

$$K_\alpha m := q^{(\lambda,\alpha)} m \text{ and } L_x m := f(\lambda, x) m$$

for $\alpha \in \Pi, x \in X$. Furthermore consider the action of $\mathcal{B}(V) \subset U_q^{\leq 0}(\mathfrak{g})$ given by the restriction of the action of $U_q(\mathfrak{g})$ on M. Using $\hat{F}_\alpha M_\lambda \subset M_{\lambda-\alpha}$ and the properties of the map f it is easy to check that these two representations satisfy the compatibility conditions from Lemma 3.4.1 and induce a representation of U on M.

Step 4: The U-coaction on $U_q(\mathfrak{g})$-modules
Let M be an integrable $U_q(\mathfrak{g})$-module. The map

$$\delta : M \to U \otimes M, \;\; \delta(m) = \sum_{\mu \geq 0} \Theta_\mu^- L_\lambda \otimes \Theta_\mu^+ m \text{ for } m \in M_\lambda$$

defines a coaction on M. Of course this map is counital. For $m \in M_\lambda$ calculate

$$\begin{aligned}
(\mathrm{id} \otimes \delta)\delta(m) &= \sum_{\nu \geq 0} \Theta_\nu^- L_\lambda \otimes \delta(\Theta_\nu^+ m) \\
&= \sum_{\mu,\nu \geq 0} \Theta_\nu^- L_\lambda \otimes \Theta_\mu^- L_\lambda K_\nu^{-1} \otimes \Theta_\mu^+ \Theta_\nu^+ m \\
&= \sum_{\mu \geq 0} \Delta(\Theta_\mu^- L_\lambda) \otimes \Theta_\mu^+ m.
\end{aligned}$$

In the last step we use the equality

$$\Delta(\Theta_\rho^-) \otimes \Theta_\rho^+ = \sum_{\substack{\nu,\mu \geq 0 \\ \nu+\mu=\rho}} \Theta_\nu^- \otimes \Theta_\mu^- K_\nu^{-1} \otimes \Theta_\mu^+ \Theta_\nu^+$$

for $\rho \geq 0$ taken from [15, 7.4], which holds in $U_q^{\leq 0}(\mathfrak{g}) \otimes U_q^{\leq 0}(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) \subset U \otimes U \otimes U_q^+(\mathfrak{g})$.

Step 5: This defines a ${}_U^U\mathcal{YD}$ structure on M
Let $m \in M_\lambda$. It suffices to check the compatibility condition for algebra generators of U. Start with the K_α:

$$\begin{aligned}
\delta(K_\alpha m) &= q^{(\lambda,\alpha)} \delta(m) \\
&= \sum_{\mu \geq 0} q^{-(\mu,\alpha)} \Theta_\mu^- L_\lambda \otimes q^{(\lambda+\mu,\alpha)} \Theta_\mu^+ m \\
&= \sum_{\mu \geq 0} K_\alpha \Theta_\mu^- L_\lambda K_\alpha^{-1} \otimes K_\alpha \Theta_\mu^+ m.
\end{aligned}$$

Then the L_x for $x \in X$:

$$\begin{aligned}\delta(L_x m) &= f(\lambda, x)\delta(m) \\ &= \sum_{\mu\geq 0} q^{(\mu,x)}\Theta_\mu^- L_\lambda \otimes f(\lambda+\mu, x)\Theta_\mu^+ m \\ &= \sum_{\mu\geq 0} L_x\Theta_\mu^- L_\lambda L_x^{-1} \otimes L_x\Theta_\mu^+ m.\end{aligned}$$

Finally consider the $F_\alpha = K_\alpha^{-1}\hat{F}_\alpha, \alpha \in \Pi$.

$$\delta(F_\alpha m) = \sum_{\mu\geq 0} \Theta_\mu^- L_{\lambda-\alpha} \otimes \Theta_\mu^+ F_\alpha m.$$

On the other hand (setting $\Theta_\mu := 0$ for $\mu \not\geq 0$)

$$\begin{aligned}& F_{\alpha_{(1)}} m_{(-1)} \mathbf{S}(F_{\alpha_{(3)}}) \otimes F_{\alpha_{(2)}} m_{(-0)} = \\ = \quad & \sum_{\mu\geq 0} F_\alpha\Theta_\mu^- L_\lambda K_\alpha \otimes K_\alpha^{-1}\Theta_\mu^+ m + \sum_{\mu\geq 0}\Theta_\mu^- L_\lambda K_\alpha \otimes F_\alpha\Theta_\mu^+ m \\ & -\sum_{\mu\geq 0}\Theta_\mu^- L_\lambda F_\alpha K_\alpha \otimes \Theta_\mu^+ m \\ = \quad & \sum_{\mu\geq 0} F_\alpha\Theta_{\mu-\alpha}^- L_\lambda K_\alpha \otimes K_\alpha^{-1}\Theta_{\mu-\alpha}^+ m + \sum_{\mu\geq 0}\Theta_\mu^- L_\lambda K_\alpha \otimes F_\alpha\Theta_\mu^+ m \\ & -\sum_{\mu\geq 0}\Theta_{\mu-\alpha}^- F_\alpha L_\lambda K_\alpha \otimes \Theta_{\mu-\alpha}^+ K_\alpha m,\end{aligned}$$

using $\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$, $\mathbf{S}(F_\alpha) = -F_\alpha K_\alpha$ and the commutation relations for the K_α's and F_α's. Now use

$$\Theta_\mu^- \otimes F_\alpha\Theta_\mu^+ + F_\alpha\Theta_{\mu-\alpha}^- \otimes K_\alpha^{-1}\Theta_{\mu-\alpha}^+ - \Theta_{\mu-\alpha}^- F_\alpha \otimes \Theta_{\mu-\alpha}^+ K_\alpha = \Theta_\mu^- \otimes \Theta_\mu^+ F_\alpha$$

for all $\mu \geq 0$ from [15, 7.1]. This yields

$$\begin{aligned}& F_{\alpha_{(1)}} m_{(-1)} \mathbf{S}(F_{\alpha_{(3)}}) \otimes F_{\alpha_{(2)}} m_{(-0)} = \\ = & \sum_{\mu\geq 0}\Theta_\mu^- L_\lambda K_\alpha \otimes \Theta_\mu^+ F_\alpha m \\ = & \sum_{\mu\geq 0}\Theta_\mu^- L_{\lambda-\alpha} \otimes \Theta_\mu^+ F_\alpha m = \delta(F_\alpha m).\end{aligned}$$

Step 6: The induced braiding is c^f
Assume that M, N are integrable $U_q(\mathfrak{g})$-modules. Let $m \in M_\lambda, n \in N_{\lambda'}$. The

braiding induced by the Yetter-Drinfeld structure defined above is

$$\begin{aligned} c_{\mathcal{YD}}(m \otimes n) &= \sum_{\mu \geq 0} \Theta_\mu^- L_\lambda n \otimes \Theta_\mu^+ m \\ &= f(\lambda', \lambda) \sum_{\mu \geq 0} \Theta_\mu^- n \otimes \Theta_\mu^+ m \\ &= c^f_{M,N}(m \otimes n). \end{aligned}$$

Remark 3.4.2. Since every $U_q(\mathfrak{g})$-linear map between integrable $U_q(\mathfrak{g})$-modules is U-linear and colinear this defines a functor from the category of integrable $U_q(\mathfrak{g})$-modules to the category ${}^U_U\mathcal{YD}$. Note that this functor preserves the braiding but is in general *not* monoidal. This is because it may happen that $L_{\lambda+\lambda'} \neq L_\lambda L_{\lambda'}$. In fact if this functor were monoidal, then c^f would satisfy the hexagon identities on every triple of integrable $U_q(\mathfrak{g})$-modules. This is not true unless the function f is a $\mathbb{Z}$-bilinear map from $\Lambda \times \Lambda$ to $k^\times$. However, if f is indeed bilinear, there is an other extension U' of $U_q^{\leq 0}(\mathfrak{g})$ and a monoidal functor from the category of integrable $U_q(\mathfrak{g})$-modules to ${}^{U'}_{U'}\mathcal{YD}$ that preserves the braiding. In this case choose $G \cong \Lambda$ identifying $\lambda \in \Lambda$ with $K_\lambda \in G$, use $L_\lambda := K_{-\lambda}$ and redo the proof above.

Remark 3.4.3. Using similar methods one can find an extension U'' of $U_q^{\geq 0}(\mathfrak{g})$ and a functor from the category of integrable $U_q(\mathfrak{g})$-modules to ${}^{U''}_{U''}\mathcal{YD}$ such that the induced braiding is $(c^f)^{-1}$. Again this functor cannot be chosen monoidal unless f is bilinear.

Remark 3.4.4. Note that U has a similar root space decomposition as $U_q^-(\mathfrak{g})$. The $\mathbb{N}$-grading of U induced by this decomposition via the height function coincides with the $\mathbb{N}$-grading induced by the Nichols algebra $\mathcal{B}(V)$. Now let M be a simple integrable $U_q(\mathfrak{g})$-module of highest weight λ and define a grading on M by

$$M(n) := \sum_{\substack{\mu \geq 0 \\ \mathrm{ht}\, \mu = n-1}} M_{\lambda-\mu}.$$

Then these gradings on M and U turn the Yetter-Drinfeld action and coaction into graded maps. If M is an arbitrary integrable $U_q(\mathfrak{g})$-module, then we can define a similar grading by decomposing M into simple submodules. With this braiding the structure maps are graded again. In particular, the braiding is a graded map.

The reduced FRT construction

Now we will determine $H^{red}(c)$ for braidings induced by finite dimensional $U_q(\mathfrak{g})$-modules. Assume that k is an algebraically closed field of characteristic zero. We will need the following proposition for Radford biproducts.

Proposition 3.4.5. Let $\psi : A \to A'$ be a morphism between Hopf algebras A, A' with bijective antipodes. Let $H \subset A, H' \subset A'$ be Hopf subalgebras with Hopf algebra projections p, p' such that the following diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "p"'] & A' \arrow[d, "p'"] \\ H \arrow[r, "\psi|H"] & H' \end{tikzcd}$$

commutes (and is well defined, i.e. $\psi(H) \subset H'$). Let $R := A^{\mathrm{co}\,p}, R' := A'^{\mathrm{co}\,p'}$ be the coinvariant subalgebras. Then $\psi(R) \subset R'$ and the diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\simeq"'] & A' \arrow[d, "\simeq"] \\ R\#H \arrow[r, "\psi|R\#\psi|H"'] & R'\#H' \end{tikzcd}$$

commutes, where the vertical isomorphisms are given by

$$A \to R\#H, a \mapsto a_{(1)}\mathbf{S}_H p(a_{(2)})\#p(a_{(3)})$$

and the corresponding map for A'.

Proof. The vertical isomorphisms are those from Radfords theorem on Hopf algebras with a projection 1.4.12. The rest of the proposition is just a computation. $\square$

Let M be a finite dimensional $U_q(\mathfrak{g})$-module with braiding $c = c^f$. Define

$$P := \{\alpha \in \Pi | E_\alpha M \neq 0\},$$

$$W := \{\lambda \in \Lambda | M_\lambda \neq 0\}.$$

Let $\tilde{G}$ be the subgroup of G generated by the $K_\lambda^{\pm 1}, \lambda \in W$. Denote by $\tilde{V}$ the subspace of V generated by the $\hat{F}_\alpha, \alpha \in P$; this is again a Yetter-Drinfeld module over G.

$$N := \{g \in \tilde{G} | \forall m \in M : gm = m\},$$

$$J := \text{k-span}\{gn - g | g \in G, n \in N \setminus \{1\}\}.$$

Theorem 3.4.6. The reduced FRT construction of (M, c^f) is given by

$$H^{red}(c^f) \simeq \mathcal{B}(\tilde{V}) \# k(\tilde{G}/N).$$

Proof. First observe that the Yetter-Drinfeld module $\tilde{V}$ over G can be restricted to a Yetter-Drinfeld module over $\tilde{G}$ because $K_\alpha \in \tilde{G}$ for all $\alpha \in P$: For $\alpha \in P$ we find $\lambda \in W, m \in M_\lambda$ with $0 \neq \hat{E}_\alpha m \in M_{\lambda+\alpha}$. By the definition of W and of the coaction on M it follows that $L_\lambda, L_{\lambda+\alpha} \in W \subset \tilde{G}$. Hence also $K_\alpha = L_{\lambda+\alpha}^{-1} L_\lambda \in \tilde{G}$.
Next we show that N acts trivially on V. Let $g \in N, \alpha \in P$; then there is an $m \in M$ with $\hat{F}_\alpha m \neq 0$ and there is an $\rho \in k$ such that $g \cdot \hat{F}_\alpha = \rho \hat{F}_\alpha$. Then

$$\rho \hat{F}_\alpha m = (g \cdot \hat{F}_\alpha) m = g \hat{F}_\alpha g^{-1} m = \hat{F}_\alpha m$$

implies $\rho = 1$ and thus $g \cdot \hat{F}_\alpha = \hat{F}_\alpha$. This means that V can be turned into a Yetter-Drinfeld module over $\tilde{G}/N$ using the canonical projection $\tilde{G} \to \tilde{G}/N$. Now we can form $\tilde{H} := \mathcal{B}(\tilde{V}) \# k(\tilde{G}/N)$. We have a canonical projection $H = \mathcal{B}(\tilde{V}) \# k\tilde{G} \to \tilde{H}$.
Now observe that the U-coaction on M can be restricted to a $\mathcal{B}(\tilde{V}) \# G$-coaction. This is possible because the $E_\alpha, \alpha \notin P$ act on M as zero. As N acts trivially on M by definition, we can turn M into a Yetter-Drinfeld module over $\tilde{H}$ using the canonical projection. We obtain then a commutative diagram of Hopf algebra projections

$$\begin{tikzcd} H(c) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \tilde{H} = \mathcal{B}(\tilde{V}) \# k(\tilde{G}/N) \arrow[dl, "\psi"] \\ H^{red}(c) & \end{tikzcd}$$

where φ is given by the universal property of $H(c)$ and π is the canonical projection. Both maps are compatible with action and coaction. To show that we have a factorization ψ we show $\ker \varphi \subset \ker \pi$: Let $x \in \ker \varphi$. Then $xM = \varphi(x)M = 0, xM^* = \varphi(x)M^* = 0$. This implies that $\pi(\ker(\varphi))$ is

a coideal of $H^{red}(c)$ that annihilates M and $M^\star$. Since $H^{red}(c)$ is $M, M^\star$-reduced we obtain $\pi(\ker(\varphi)) = 0$.
To see that ψ is injective we will first show that all occurring maps are graded. In Remark 3.4.4 we saw that M has a $\mathbb{N}$-grading such that the structure maps of the Yetter-Drinfeld module structure over U are graded. This grading turns the $\tilde{H}$ action and coaction into graded maps. Thus $H(c)$ and $H^{red}(c)$ have $\mathbb{Z}$-gradings such that the projection π, the actions and the coactions are graded (This can easily be seen in the construction of $H(c)$ given in [39]: Start with a homogenous basis $m_1, \ldots, m_r$ of M and grade $H(c)$ by giving the generator T_{ij} the degree $deg(m_i) - deg(m_j)$). Using the compatibility condition between φ and the $H(c)$ resp. $\tilde{H}$-coactions it is easy to see that also φ is a graded map. Then by construction also the map ψ is graded. It follows that $H^{red}(c)$ is actually $\mathbb{N}$-graded.
Now both $\tilde{H}$ and $H^{red}(c)$ are graded Hopf algebras, hence admit Hopf algebra projections onto the zeroth components. As ψ is a graded map we can apply Proposition 3.4.5 to our situation. So to show that ψ is injective it suffices to show that $\psi|k(\tilde{G}/N)$ and $\psi|\mathcal{B}(\tilde{V})$ are injective.
First show that $\psi|k(\tilde{G}/N)$ is injective: Let $\bar{x}, \bar{y} \in \tilde{G}/N$ such that $\psi(\bar{x}) = \psi(\bar{y})$. This means $xm = ym$ for all $m \in M$ and thus $xy^{-1} \in N$. Hence $\bar{x} = \bar{y}$, showing that $\psi|\tilde{G}/N$ is injective. The claim follows by linear algebra.
On the other hand, let I be the kernel of $\psi|\mathcal{B}(\tilde{V})$. As this is a graded morphism of algebras and coalgebras, I is a coideal and an ideal generated by homogeneous elements. By the characterization of Nichols algebras from [5], $I = 0$ if $I \cap \tilde{V} = 0$ (i.e. I is generated by elements of degree ≥ 2).
So assume we have $x \in I \cap \tilde{V}$ and write $x = \sum_{\alpha \in P} r_\alpha \hat{F}_\alpha$ for scalars $r_\alpha \in k$. Then $xM = 0$, as $I \subset \ker \psi$. The weight-space grading of the module M yields that for all $\alpha \in P$

$$r_\alpha \hat{F}_\alpha M = 0.$$

For $\alpha \in P$ we have $E_\alpha M \neq 0$ and hence also $\hat{F}_\alpha M \neq 0$. This implies $r_\alpha = 0$ for all $\alpha \in P$ and thus $x = 0$. □

Remark 3.4.7. The set P is a union of connected components of the Coxeter graph of $\mathfrak{g}$.
In particular if $\mathfrak{g}$ is simple, we have $\tilde{V} = V$ and thus $H^{red}(c)$ is obtained from U just by dividing out the ideal generated by the set

$$\{g - h | g, h \in G, \forall m \in M : gm = hm\}.$$

In the general case we obtain that $H^{red}(c)$ may be viewed as the "non-positive part of a quantized enveloping algebra of $\hat{\mathfrak{g}}$" (where $\hat{\mathfrak{g}}$ is the Lie subalgebra of $\mathfrak{g}$ generated by the $E_\alpha, H_\alpha, F_\alpha, \alpha \in P$) in the sense that $H^{red}(c)$ is a biproduct of the negative part $U_q^-(\hat{\mathfrak{g}})$ with a finitely generated abelian group.

Proof. The second part of the remark follows from the proof of the theorem above. We show only that P is a union of connected components of the Coxeter graph, i.e. if $\alpha \in P$ and $\beta \in \Pi$ with $(\alpha, \beta) < 0$ then also $\beta \in P$.
So assume we have $\alpha \in P, \beta \in \Pi$ such that $(\alpha, \beta) < 0$. Thus we have $E_\alpha M \neq 0$ and we will show $E_\beta M \neq 0$. Let $\lambda \in \Lambda, m \in M_\lambda$ with $E_\alpha m \neq 0$. If $(\lambda, \beta) = 0$ replace m by $E_\alpha m$ and λ by $\lambda + \alpha$. Hence $0 \neq m \in M_\lambda$ and $(\lambda, \beta) \neq 0$. Let $U_q(\mathfrak{sl}_2)_\beta$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $E_\beta, F_\beta, K_\beta$ and K_β^{-1}; it is isomorphic to $U_{q^{(\beta,\beta)}}(\mathfrak{sl}_2)$ as a Hopf algebra. Consider the $U_q(\mathfrak{sl}_2)_\beta$ submodule N of M generated by m. If $E_\beta m \neq 0$ we have $\beta \in P$ and the proof is done. So assume $E_\beta m = 0$, hence m is a highest weight vector for the $U_q(\mathfrak{sl}_2)_\beta$-module N. As $(\lambda, \beta) \neq 0$, N is not one-dimensional. Thus we have $E_\beta(F_\beta m) \neq 0$, implying $\beta \in P$. $\square$

Remark 3.4.8. It is an open question if there is a combinatorial description of those triangular braidings for which the reduced FRT construction is generated by group-like and skew-primitive elements.

Chapter 4

Nichols algebras of $U_q(\mathfrak{g})$-modules

One motivating example of triangular braidings are those braidings induced by the quasi-$\mathcal{R}$-matrix of a deformed enveloping algebra $U_q(\mathfrak{g})$. In particular Andruskiewitsch [1] raised the question on the structure of the Nichols algebras of these modules. Apart from cases when the braidings are of Hecke type (see [38] and [5]) nothing seemed to be known in this area.
These algebras are by definition bialgebras of triangular type and we already considered some special examples in Section 2.6 for the case that $\mathfrak{g} = \mathfrak{sl}_2$. Nevertheless for a general study of more complicated Lie algebras and higher-dimensional modules the combinatorial method from Chapter 2 does not seem to be suitable. In this chapter we present a second approach which is motivated by the work of Rosso [38], but takes a different point of view. Rosso uses the knowledge on Nichols algebras of Hecke type to obtain information on the structure of the nonnegative parts of the deformed enveloping algebras; we will obtain new results for the Nichols algebras by applying knowledge on the deformed enveloping algebras. With this approach we get new results on Nichols algebras also in cases when the braiding is not of Hecke type.
In Section 3.4 we have realized the braidings on $U_q(\mathfrak{g})$-modules M as Yetter-Drinfeld braidings over a Hopf algebra of the form $U = \mathcal{B}(V)\#kG$, where V is a braided vector space with diagonal braiding and G is a free abelian group. An important observation for our method is that the braided biproduct $\mathcal{B}(M)\#\mathcal{B}(V)$ is again a Nichols algebra of a braided vector space with diagonal braiding. Our results on braided biproducts actually hold in a more general setting where the base Hopf algebra is not necessarily an abelian group algebra. In Sections 4.1, 4.2 and 4.3 we will present these general results. In Section 4.4 we determine those $U_q(\mathfrak{g})$-modules that lead to Nichols algebras of finite Gelfand-Kirillov dimension; Rosso considered only some

special cases of the simple modules we list in Table 4.1. In Section 4.5 we calculate the defining relations of the Nichols algebras of $U_q(\mathfrak{g})$-modules, even if the braiding is not necessarily of Hecke type.
When we deal with braided biproducts we will use different types of Sweedler notation according to the following convention.

Notation 4.0.9. Assume that H is a Hopf algebra with bijective antipode, R a Hopf algebra in ${}^H_H\mathcal{YD}$ such that $R\#H$ has bijective antipode and let Q be a braided Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$. In this chapter the following conventions for Sweedler notation are used:

1. The Sweedler indices for the comultiplication in usual Hopf algebras are lower indices with round brackets: $\Delta_H(h) = h_{(1)} \otimes h_{(2)}$.
2. The Sweedler indices for the comultiplication in braided Hopf algebras in ${}^H_H\mathcal{YD}$ are upper indices with round brackets: $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$.
3. The Sweedler indices for the comultiplication in braided Hopf algebras in ${}^{R\#H}_{R\#H}\mathcal{YD}$ are upper indices with square brackets: $\Delta_Q(x) = x^{[1]} \otimes x^{[2]}$.
4. For H-coactions we use lower Sweedler indices with round brackets: $\delta_H(v) = v_{(-1)} \otimes v_{(0)}$.
5. For $R\#H$-coactions we use lower Sweedler indices with square brackets: $\delta_{R\#H}(m) = m_{[-1]} \otimes m_{[0]}$.

4.1 Braided biproducts

In this section a braided version of Radfords biproduct construction is introduced. This is done for arbitrary braided categories in [6]. Here an ad-hoc approach for the category ${}^H_H\mathcal{YD}$ is presented, that leads very quickly to the necessary results. Let H be a Hopf algebra with bijective antipode and R a Hopf algebra in ${}^H_H\mathcal{YD}$ such that $R\#H$ has bijective antipode. Moreover let Q be a Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$. Consider the projection of Hopf algebras

$$\varepsilon \otimes \varepsilon \otimes H : Q\#(R\#H) \to H.$$

Proposition 4.1.1. The space of (right) coinvariants with respect to the projection $\varepsilon \otimes \varepsilon \otimes \mathrm{id}_H$ is $Q \otimes R \otimes 1$.

Proof. One inclusion is trivial. So assume there is a coinvariant

$$T = \sum_{i=1}^{r} x_i \# r_i \# h_i \in (Q\#(R\#H))^{\mathrm{co}\,\varepsilon\otimes\varepsilon\otimes H}.$$

The $x_i \otimes r_i$ can be chosen linearly independent. Using the formulas for the comultiplication of the Radford biproduct one obtains

$$T \otimes 1_H = (\mathrm{id}_Q \otimes \mathrm{id}_R \otimes \mathrm{id}_H \otimes \varepsilon \otimes \varepsilon \otimes \mathrm{id}_H)\Delta(T) = \sum_{i=1}^{r} x_i \otimes r_i \otimes h_{i_{(1)}} \otimes h_{i_{(2)}}.$$

This implies $h_i = \varepsilon(h_i)1$ for all $1 \leq i \leq r$ and thus $T \in Q\#R\#1$. □

Definition 4.1.2. $Q \otimes R$ inherits the structure of a Hopf algebra in ${}^H_H\mathcal{YD}$ from the coinvariants. This object is called the *braided biproduct of* Q *and* R and is denoted by $Q\#R$.

Q is a subalgebra of $Q\#R$ (via the inclusion $x \mapsto x\#1$) and R is a braided Hopf subalgebra of $Q\#R$.
Note that $Q \in {}^H_H\mathcal{YD}$ via the inclusion $H \to R\#H$ and the projection

$$\pi_H : R\#H \to H,\ r\#h \mapsto \varepsilon(r)h.$$

However Q is in general not a braided Hopf algebra in ${}^H_H\mathcal{YD}$.
By construction of $Q\#R$ it is obvious that

$$Q\#(R\#H) \simeq (Q\#R)\#H,\ x\#(r\#h) \mapsto (x\#r)\#h.$$

Structure maps

The following list contains formulas for the structure maps of $Q\#R$. The proofs are left to the reader. For all $x, x' \in Q, r, r' \in R, h \in H$:

$$\begin{aligned}
(x\#r)(x'\#r') &= x\left[(r^{(1)}\#r^{(2)}{}_{(-1)}) \cdot x'\right]\#r^{(2)}{}_{(0)}r', \\
\Delta_{Q\#R}(x\#r) &= x^{[1]}\#\theta_R(x^{[2]}{}_{[-2]})\left[\pi_H(x^{[2]}{}_{[-1]}) \cdot r^{(1)}\right] \otimes x^{[2]}{}_{[0]}\#r^{(2)}, \\
\delta_H(x\#r) &= \pi_H(x_{[-1]})r_{(-1)} \otimes x_{[0]}\#r_{(0)}, \\
h \cdot (x\#r) &= \Big((1\#h_{(1)}) \cdot x\Big)\#h_{(2)} \cdot r.
\end{aligned}$$

Here $\theta_R = \mathrm{id}_R \otimes \varepsilon_H : R\#H \to R$ and $\pi_H = \varepsilon_R \otimes \mathrm{id}_H : R\#H \to H$ are the maps from Subsection 1.4.2. Note that the action and coaction correspond to the tensor product of Yetter-Drinfeld modules over H.

The braided adjoint action

For any Hopf algebra R in ${}^H_H\mathcal{YD}$ the braided adjoint action is defined by

$$\mathrm{ad}_c : R \to \mathrm{End}(R),\ \mathrm{ad}_c(r)(r') := r^{(1)} \left({r^{(2)}}_{(-1)} \cdot r' \right) \mathbf{S}_R \left({r^{(2)}}_{(0)} \right).$$

In the usual Radford biproduct $R\#H$ the following rules are valid:

$$\begin{aligned} \mathrm{ad}(1\#h)(1\#h') &= 1\#\mathrm{ad}(h)(h'), \\ \mathrm{ad}(1\#h)(r\#1) &= (h \cdot r)\#1, \\ \mathrm{ad}(r\#1)(r'\#1) &= \mathrm{ad}_c(r)(r')\#1 \end{aligned}$$

for all $r, r' \in R, h, h' \in H$.
In the braided biproduct $Q\#R$ the corresponding rules

$$\begin{aligned} \mathrm{ad}_c(1\#r)(1\#r') &= 1\#\mathrm{ad}_c(r)(r'), \\ \mathrm{ad}_c(1\#r)(x\#1) &= ((r\#1) \cdot x)\#1, \\ \mathrm{ad}_c(x\#1)(x'\#1) &= \mathrm{ad}_c(x)(x')\#1 \end{aligned}$$

hold for all $x, x' \in Q, r, r' \in R$. Note that in the last equation on the right side the ${}^{R\#H}_{R\#H}\mathcal{YD}$ structure on Q is used to define ad_c.

4.2 Graded Yetter-Drinfeld modules

For this section assume that $A = \oplus_{n\geq 0} A(n)$ is a graded Hopf algebra with bijective antipode. Then $H := A(0)$ is a Hopf algebra with bijective antipode. In this section the notion of a graded Yetter-Drinfeld modules over A is defined. This class of Yetter-Drinfeld modules is the natural context for the extension Theorem 4.3.1.

Definition 4.2.1. M is called a *graded Yetter-Drinfeld module* (over A) if $M \in {}^A_A\mathcal{YD}$ and it has a grading $M = \oplus_{n\geq 1} M(n)$ as a vector space such that the action and the coaction are graded maps with respect to the usual grading on tensor products

$$(A \otimes M)(n) = \sum_{i+j=n} A(i) \otimes M(j).$$

The subspace $M_H := \{m \in M | \delta(m) \in H \otimes M\}$ is called the *space of highest weight vectors* of M.
M is said to be *of highest weight* if it is a graded Yetter-Drinfeld module, $M_H = M(1)$ and M is generated by M_H as an A-module.

Lemma 4.2.2. Let $M \in {}^A_A\mathcal{YD}$ be of highest weight. The space of highest weight vectors M_H of M is a Yetter-Drinfeld module over H with action and coaction given by the restrictions of the structure maps on M.

Proof. M_H is an H submodule by the Yetter-Drinfeld condition. To see that M_H is a H-comodule fix a basis $(h_i)_{i\in I}$ of H. There are scalars $(\alpha^i_{jl})_{i,j,l\in I}$ such that for all $i \in I$

$$\Delta(h_i) = \sum_{j,l\in I} \alpha^i_{jl} h_j \otimes h_l.$$

Furthermore let $m \in M_H$. Now there are elements $(m_i)_{i\in I}$ of M (almost all equal to zero) such that

$$\delta(m) = \sum_{i\in I} h_i \otimes m_i.$$

It suffices to show that $m_j \in M_H$ for all $j \in I$. We have

$$\sum_{j\in I} h_j \otimes \delta(m_j) = (H \otimes \delta)\delta(m) = (\Delta \otimes M)\delta(m) = \sum_{i,j,l\in I} \alpha^i_{jl} h_j \otimes h_l \otimes m_i$$

and thus for all $j \in I$

$$\delta(m_j) = \sum_{i,l\in I} \alpha^i_{jl} h_l \otimes m_i \in H \otimes M,$$

showing $m_j \in M_H$ for all $j \in I$. □

Example 4.2.3. Assume that $\operatorname{char} k = 0$ and that $q \in k$ is not a root of unity. Let U be the extension of $U_q^{\leq 0}(\mathfrak{g})$ defined in Section 3.4. By Remark 3.4.4 the Yetter-Drinfeld module structure defined in Section 3.4 makes M a graded Yetter-Drinfeld module over U. It is easy to see that it is a graded Yetter-Drinfeld module of highest weight. Moreover the space of highest weight vectors is exactly the space spanned by the vectors that are of highest weight in the usual sense.

The next step is to extend the grading from graded Yetter-Drinfeld modules to their Nichols algebras. In general the coradical grading of the Nichols algebra does not turn the action and coaction into graded maps.

Proposition 4.2.4. Let $M \in {}^A_A\mathcal{YD}$ be a graded Yetter-Drinfeld module. Then there is a grading

$$\mathcal{B}(M) = \bigoplus_{n\geq 0} \mathcal{B}(M)[n],$$

turning $\mathcal{B}(M)$ into a graded Yetter-Drinfeld module over A and into a graded braided Hopf algebra such that

$$\mathcal{B}(M)[0] = k1 \text{ and } \mathcal{B}(M)[1] = M(1).$$

Proof. Grade the tensor algebra $T(M)$ by giving $M(n)$ the degree n. Then the action and the coaction are graded and so is the braiding. Thus the quantum symmetrizer maps are graded maps. As the kernel of the projection $T(M) \to \mathcal{B}(M)$ is just the direct sum of the kernels of the quantum symmetrizers [40], it is a graded Hopf ideal. So the quotient $\mathcal{B}(M)$ admits the desired (induced) grading. □

4.3 Braided biproducts of Nichols algebras

In this section the results of the preceding sections are specialized to a braided biproduct of two Nichols algebras. The next theorem is a generalization of [38, Proposition 2.2] from abelian group algebras to arbitrary Hopf algebras H with bijective antipode. This result allows to reduce the study of Nichols algebras of graded Yetter-Drinfeld modules over $\mathcal{B}(V)\#H$ to the study of Yetter-Drinfeld modules over H.

Theorem 4.3.1. Assume that H is a Hopf algebra with bijective antipode, $V \in {}^H_H\mathcal{YD}$ and set $A := \mathcal{B}(V)\#H$ as a graded Hopf algebra with grading $A(n) := \mathcal{B}(V)(n)\#H$. Furthermore let $M \in {}^A_A\mathcal{YD}$ be a graded Yetter-Drinfeld module. If M is of highest weight then there is an isomorphism of graded braided Hopf algebras in ${}^H_H\mathcal{YD}$

$$\phi : \mathcal{B}(M)\#\mathcal{B}(V) \to \mathcal{B}(M_H \oplus V)$$

such that for all $m \in M_H, v \in V$ we have $\phi(m\#1) = m$ and $\phi(1\#v) = v$. Here the left side is graded by the tensor product grading and the grading for $\mathcal{B}(M)$ is taken from Proposition 4.2.4.

Proof. $B := \mathcal{B}(M)\#\mathcal{B}(V)$ is graded as a braided Hopf algebra: All the structure maps of B are obtained from the structure maps of $H, V, \mathcal{B}(V), M$ and $\mathcal{B}(M)$. As all these maps are graded (giving V the degree 1 and H the degree 0) this part is done.
Next check that $B[1] = P(B)$. As B is graded as a coalgebra and $B[0] = k1$ it is clear that $B[1] \subset P(B)$. To show the other inclusion identify B with the coinvariant subalgebra in $\mathcal{B}(M)\#(\mathcal{B}(V)\#H)$. By construction of the grading

$$B[1] = M_H\#1\#1 \oplus 1\#V\#1.$$

For a primitive element $t = \sum_{i=0}^{r} x_i \# u_i \# 1 \in P(B)$, the coproduct of t is given by

$$\Delta(t) = \sum_{i=0}^{r} x_i^{[1]} \# {x_i^{[2]}}_{[-2]} \left(u_i^{(1)} \# 1\right) \iota \mathbf{S} \pi \left({x_i^{[2]}}_{[-1]}\right) \otimes {x_i^{[2]}}_{[0]} \# u_i^{(2)} \# 1,$$

where π denotes the Hopf algebra projection from $A = \mathcal{B}(V)\#H$ onto H and ι is the inclusion of H into $\mathcal{B}(V)\#H$.
The x_i can be chosen linearly independent and such that $\varepsilon(x_i) \neq 0$ if and only if $i = 0$. Applying the map $\mathrm{id}_{\mathcal{B}(M)} \# \mathrm{id}_{\mathcal{B}(V)} \otimes \mathrm{id}_H \otimes \varepsilon \# \mathrm{id}_{\mathcal{B}(V)} \otimes \mathrm{id}_H$ to the equality $1 \otimes t + t \otimes 1 = \Delta(t)$ yields

$$u_0 \in P(\mathcal{B}(V)) = V \text{ and } \forall 1 \leq i \leq r : u_i \in k1.$$

This means $t = x\#1\#1 + 1\#u\#1$ for $x \in \mathcal{B}(M), u \in V$. In particular $x\#1\#1 \in P(B)$. Now calculate

$$\Delta(x\#1\#1) = x^{[1]} \# {x^{[2]}}_{[-2]} \iota \mathbf{S} \pi \left({x^{[2]}}_{[-1]}\right) \otimes {x^{[2]}}_{[0]} \# 1 \# 1.$$

Consider the equality $x\#1\#1 \otimes 1\#1\#1 + 1\#1\#1 \otimes x\#1\#1 = \Delta(x\#1\#1)$ and apply first the map $\mathrm{id}_{\mathcal{B}(M)} \otimes \varepsilon \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{B}(M)} \otimes \varepsilon \otimes \varepsilon$. This yields $x \in P(\mathcal{B}(M)) = M$. Then apply the map $\varepsilon \otimes \mathrm{id}_{\mathcal{B}(V)} \otimes \mathrm{id}_H \otimes \mathrm{id}_{\mathcal{B}(M)} \otimes \varepsilon \otimes \varepsilon$. This yields

$$x_{[-2]} \iota \mathbf{S} \pi \left(x_{[-1]}\right) \otimes x_{[0]} = 1 \otimes x,$$

implying that $\delta(x) = \iota\pi(x_{[-1]}) \otimes x_{[0]} \in H \otimes M$ and thus $x \in M_H$.
It remains to show that B is actually generated by $B[1]$. Of course B is generated by $\mathcal{B}(M)\#1\#1$ and $1\#\mathcal{B}(V)\#1$. So it suffices to show that $M\#1\#1$ is contained in the subalgebra generated by $M_H\#1\#1$ and $1\#V\#1$. As M is of highest weight it is generated as a $\mathcal{B}(V)\#H$-module by M_H. Using that M_H is an H-module this means

$$\begin{aligned} M &= (\mathcal{B}(V)\#H) \cdot M_H = ((\mathcal{B}(V)\#1)(1\#H)) \cdot M_H \\ &= (\mathcal{B}(V)\#1) \cdot M_H = \mathrm{ad}_c(\mathcal{B}(V))(M_H)\#1. \end{aligned}$$

Thus within B, M is generated by M_H under the braided adjoint action of $\mathcal{B}(V)$. All together M_H and V generate B. $\square$

4.4 The Gelfand-Kirillov dimension of Nichols algebras of finite-dimensional $U_q(\mathfrak{g})$-modules

If q is not a root of unity we do not expect that the Nichols algebras of integrable $U_q(\mathfrak{g})$-modules are finite-dimensional. If we want to consider the size of the Nichols algebras, the Gelfand-Kirillov dimension is the right invariant. We will first collect some basic statements.

Definition 4.4.1. [22] Let $A = \oplus_{n\geq 0} A(n)$ be a graded algebra which is generated by $A(1)$. For all $n \in \mathbb{N}$ define $d(n) := \dim \oplus_{0\leq i\leq n} A(i)$. The *Gelfand-Kirillov dimension* of A is

$$\text{GK-dim}\, A = \inf\{\rho \in \mathbb{R} | d(n) \leq n^\rho \text{ for almost all } n \in \mathbb{N}\}.$$

Remark 4.4.2. Let A be as in the definition.

1. The Gelfand-Kirillov dimension of A does not depend on the grading of A. Furthermore it can be defined in a similar way for arbitrary algebras.

2. Assume there are polynomials $p, \tilde{p} \in \mathbb{R}[X]$ of degree r such that for almost all $n \in \mathbb{N}$ we have
$$p(n) \leq d(n) \leq \tilde{p}(n).$$
Then the Gelfand-Kirillov dimension of A is r.

3. Assume that A has a PBW basis $(S, \leq, h)$ with finite set S made up of homogenous elements and $h(s) = \infty$ for all $s \in S$. Then the Gelfand-Kirillov dimension of A is the cardinality of S.

4. Assume that A has a PBW basis $(S, \leq, h)$ with infinite set S made up of homogenous elements and $h(s) = \infty$ for all $s \in S$. Then the Gelfand-Kirillov dimension of A is infinite.

Proof. For part one our reference is [22, Chapter 1 and 2]. Part 2: First let $\rho > r$. Then $\frac{\tilde{p}(n)}{n^\rho}$ tends to zero for growing n. Thus we have $d(n) \leq \tilde{p}(n) \leq n^\rho$ for almost all $n \in \mathbb{N}$ and GK-dim $A \leq \rho$. This implies GK-dim $A \leq r$. On the other hand let $\rho < r$. Then $\frac{p(n)}{n^\rho}$ tends to infinity for large n. Thus we have $d(n) \geq p(n) > n^\rho$ for almost all $n \in \mathbb{N}$. This implies GK-dim $A \geq r$. For part 3 assume that we have a PBW basis as in the second part of the remark. Denote the elements of S by $s_1, \dots, s_r$ with $s_1 < \ldots < s_r$ and let d_i be the degree of s_i for all $1 \leq i \leq r$. Obviously the elements of the form

$$s_1^{e_1} \ldots s_r^{e_r}$$

with $e_1, \ldots, e_r \in \mathbb{N}_0$ and $\sum_{i=1}^r e_i d_i \leq n$ form a basis of $\oplus_{i=1}^n A(i)$. So by 2) it suffices to find polynomials $p, q \in \mathbb{R}[X]$ of degree r such that for almost all $n \in \mathbb{N}$

$$p(n) \leq \left| \left\{ (e_1, \ldots, e_r) \in \mathbb{N}_0^r | \sum_{i=1}^r e_i d_i \leq n \right\} \right| \leq q(n).$$

This follows by an easy geometric argument. The proof of part 4 is similar to that of part 2. □

Assume that char $k = 0$ and that $q \in k$ is not a root of unity. For this section let M be a finite-dimensional integrable $U_q(\mathfrak{g})$-module with braiding c^f as in Example 1.3.7. Moreover assume that the root system of $\mathfrak{g}$ is normalized as in Subsection 1.2.3. Recall that $\lambda \in \Lambda$ is called a *highest weight* of M if there exists a $0 \neq m \in M_\lambda$ such that for all $\alpha \in \Pi : E_\alpha m = 0$. The first result will be a criterion to decide whether $\mathcal{B}(M)$ has finite Gelfand-Kirillov dimension or not.
From now on we will restrict to braidings of a special form. This restriction is necessary due to missing information on Nichols algebras of diagonal type.

Definition 4.4.3. The braiding c^f is *of exponential type with function* φ if the map $f : \Lambda \times \Lambda \to k^\times$ is of the form

$$f(\lambda, \mu) = v^{-d\varphi(\lambda,\mu)}$$

for some $v \in k$, $d \in 2\mathbb{Z}$ such that $v^d = q$ and for a map $\varphi : \Lambda \times \Lambda \to \frac{2}{d}\mathbb{Z}$ with the property that for $\lambda, \lambda' \in \Lambda, \nu \in \mathbb{Z}\Phi$

$$\varphi(\lambda + \nu, \lambda') = \varphi(\lambda, \lambda') + (\nu, \lambda') \text{ and } \varphi(\lambda, \lambda' + \nu) = \varphi(\lambda, \lambda') + (\lambda, \nu).$$

c^f is *of strong exponential type* if it is of exponential type with a function φ such that for every highest weight λ of M with $\varphi(\lambda, \lambda) \leq 0$ we have

$$\varphi(\lambda, \lambda) = 0$$

and for every other highest weight λ' of M

$$\varphi(\lambda, \lambda') + \varphi(\lambda', \lambda) = 0.$$

As shown in Example 4.2.3 the module M is a graded Yetter-Drinfeld module of highest weight over U (the grading on $U = \mathcal{B}(V)\#kG$ is the one induced by the Nichols algebra). Assume that $M = \oplus_{1 \leq i \leq r} M_i$ is the decomposition of the $U_q(\mathfrak{g})$-module M into irreducible submodules. For all $1 \leq i \leq r$ choose a highest weight vector $m_i \in M_i$ and denote by $\lambda_i \in \Lambda$ the weight of m_i.

Then the space $M_{kG} \oplus V$ has basis $\{m_1, \ldots, m_r\} \cup \{\hat{F}_\alpha | \alpha \in \Pi\}$. If c^f is of exponential type, the braiding on $M_{kG} \oplus V$ is given by

$$\begin{aligned} c(\hat{F}_\alpha \otimes \hat{F}_\beta) &= v^{-d(\beta,\alpha)} \hat{F}_\beta \otimes \hat{F}_\alpha, \\ c(\hat{F}_\alpha \otimes m_j) &= v^{d(\lambda_j,\alpha)} m_j \otimes \hat{F}_\alpha, \\ c(m_i \otimes \hat{F}_\beta) &= v^{d(\beta,\lambda_i)} \hat{F}_\beta \otimes m_i \text{ and} \\ c(m_i \otimes m_j) &= f(\lambda_j, \lambda_i) m_j \otimes m_i = v^{-d\varphi(\lambda_i,\lambda_j)} m_j \otimes m_i. \end{aligned}$$

Let $P := \Pi \dot{\cup} \{1, \ldots, r\}$. If c^f is of strong exponential type there is always a matrix $(b_{ij})_{i,j \in P} \in \mathbb{Q}^{P \times P}$ such that the following conditions are satisfied:

$$\begin{aligned} \forall \alpha, \beta \in \Pi : &\quad 2(\alpha, \beta) = (\alpha, \alpha) b_{\alpha\beta} && (1) \\ \forall \alpha \in \Pi, 1 \le i \le r : &\quad 2(\alpha, \lambda_i) = -(\alpha, \alpha) b_{\alpha i} && (2) \\ \forall \alpha \in \Pi, 1 \le i \le r : &\quad 2(\alpha, \lambda_i) = -\varphi(\lambda_i, \lambda_i) b_{i\alpha} && (3) \\ \forall 1 \le i, j \le r : &\quad \varphi(\lambda_i, \lambda_j) + \varphi(\lambda_j, \lambda_i) = \varphi(\lambda_i, \lambda_i) b_{ij} && (4) \\ \forall 1 \le i \le r, i \ne j \in P, \alpha \in \Pi : &\quad \varphi(\lambda_i, \lambda_i) = 0 \Rightarrow b_{ii} = 2, b_{ij} = 0, b_{i\alpha} = 0 && (5) \end{aligned}$$

The matrix $(b_{ij})_{i,j \in P}$ will be called the *extended Cartan matrix* of M.

Theorem 4.4.4. Let k be an algebraically closed field of characteristic zero, $q \in k$ not a root of unity. Assume that the braiding c^f on the finite-dimensional integrable $U_q(\mathfrak{g})$-module M is of exponential type with a symmetric function φ (i.e. $\varphi(\lambda, \lambda') = \varphi(\lambda', \lambda)$ for all $\lambda, \lambda' \in \Lambda$).
Then the Nichols algebra $\mathcal{B}(M, c^f)$ has finite Gelfand-Kirillov dimension if and only if c^f is of strong exponential type (with function φ) and the extended Cartan matrix (b_{ij}) is a Cartan matrix of finite type.

Proof. Denote the basis of $M_{kG} \oplus V$ by $x_i, i \in P$ where $x_\alpha := \hat{F}_\alpha$ and $x_i := m_i$ for $\alpha \in \Pi$ and $1 \le i \le r$. The braiding of $M_{kG} \oplus V$ is of the form

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \ \forall i, j \in P,$$

where the q_{ij} can be read off the formulas given above:

$$q_{\alpha\beta} = v^{-d(\beta,\alpha)}, q_{\alpha i} = v^{d(\alpha,\lambda_i)}, q_{i\alpha} = v^{d(\alpha,i)} \text{ and } q_{ij} = v^{-d\varphi(\lambda_i,\lambda_j)}$$

for all $\alpha, \beta \in \Pi$ and $1 \le i, j \le r$.
The if-part: By the definition of (b_{ij}) for all $i, j \in P$

$$q_{ij} q_{ji} = {q_{ii}}^{b_{ij}}.$$

For all $\alpha \in \Pi$ and for all $1 \le i \le r$ with $\varphi(\lambda_i, \lambda_i) \ne 0$ define

$$d_\alpha := \frac{d(\alpha, \alpha)}{2} \text{ and } d_i := \frac{d\varphi(\lambda_i, \lambda_i)}{2}$$

and for $1 \leq i \leq r$ with $\varphi(\lambda_i, \lambda_i) = 0$ define $d_i := 1$. These (d_i) are positive integers satisfying

$$d_i b_{ij} = d_j b_{ji} \ \forall i, j \in P.$$

Because φ is symmetric one gets for all $i, j \in P$

$$q_{ij} = v^{-d_i b_{ij}}.$$

This means that the braiding on $M_{kG} \oplus V$ is of Frobenius-Lusztig type with generalized Cartan matrix (b_{ij}). As (b_{ij}) is a finite Cartan matrix, $\mathcal{B}(M_{kG} \oplus V)$ has finite Gelfand-Kirillov dimension by [4, Theorem 2.10.]. Thus the subalgebra $\mathcal{B}(M)$ has finite Gelfand-Kirillov dimension [22, Lemma 3.1.].
The only-if-part: By 2.2.4 $\mathcal{B}(M)$ has a PBW basis and because the Gelfand-Kirillov dimension is finite the set of PBW generators S_M must be finite. Similarly $\mathcal{B}(V)$ has a PBW basis and because it has finite Gelfand-Kirillov dimension (see [4, Theorem 2.10.]) its set of PBW generators S_V is also finite. So the finite set

$$S := \{s\#1 | s \in S_M\} \dot{\cup} \{1\#s' | s' \in S_V\}$$

forms a set of PBW generators for $\mathcal{B}(M)\#\mathcal{B}(V) \cong \mathcal{B}(M_{kG} \oplus V)$. This implies that $\mathcal{B}(M_{kG} \oplus V)$ has finite Gelfand-Kirillov dimension. Now [38, Lemma 14 and 20] allows us to find integers $c_{ij} \leq 0, i, j \in P$ such that

$$q_{ij} q_{ji} = q_{ii}^{c_{ij}} \ \forall i, j \in P.$$

Using the definition of the q_{ij} one obtains that the c_{ij} must satisfy the equations $(1) - (4)$ from the definition of (b_{ij}) with $b_{ij}, i, j \in P$ replaced by $c_{ij}, i, j \in P$. Because of relations (3) and (4), φ must satisfy the condition from the definition of strong exponential braidings. Furthermore one may assume $c_{ii} = 2$ for all $i \in P$ and $c_{ij} = 0$ for all $1 \leq i \leq r$ with $\varphi(\lambda_i, \lambda_i) = 0, i \neq j \in P$. This means that $b_{ij} = c_{ij}$ for all $i, j \in P$. Now observe that b_{ij} is a generalized Cartan matrix. Exactly as in the "only-if" part of the proof the braiding in $M_{kG} \oplus V$ is of Frobenius-Lusztig type with generalized Cartan matrix (b_{ij}). By [4, Theorem 2.10.] (b_{ij}) is a finite Cartan matrix because $\mathcal{B}(M_{kG} \oplus V)$ has finite Gelfand-Kirillov dimension. □

Explicit calculations for simple $U_q(\mathfrak{g})$-modules

Now the results above are used to determine for each finite-dimensional simple complex Lie algebra $\mathfrak{g}$ all pairs (λ, φ) such that the Nichols algebra of the $U_q(\mathfrak{g})$-module of highest weight λ together with the braiding defined by the function φ has finite Gelfand-Kirillov dimension. First observe that (as only modules of highest weight are considered) one may assume that the function φ is of the form

$$\varphi(\mu,\nu) = (\mu,\nu) + x \text{ for } \mu,\nu \in \Lambda$$

for some $x \in \mathbb{Q}$. This is true because the braiding c^f depends only on the values $\varphi(\lambda',\lambda'')$ for those weights $\lambda',\lambda'' \in \Lambda$ such that $M_{\lambda'} \neq 0, M_{\lambda''} \neq 0$. They are all in the same coset of $\mathbb{Z}\Phi$ in Λ. Thus one can choose

$$x := \varphi(\lambda,\lambda) - (\lambda,\lambda)$$

for any weight $\lambda \in \Lambda$ with $M_\lambda \neq 0$.

Theorem 4.4.5. Assume that k is an algebraically closed field of characteristic zero and that $q \in k$ is not a root of unity. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra with weight lattice Λ. Fix a $U_q(\mathfrak{g})$-module M of highest weight $\lambda \in \Lambda$ and a value $x \in \mathbb{Q}$. Let d' be the least common multiple of the denominator of x and the determinant of the Cartan matrix of $\mathfrak{g}$. Let $d := 2d'$ and fix $v \in k$ with $v^d = q$. Define a function

$$f : \Lambda \times \Lambda \to k^\times,\ (\lambda,\lambda') \mapsto v^{d((\lambda,\lambda')+x)}.$$

Then the Nichols algebra $\mathcal{B}(M, c^f_{M,M})$ has finite Gelfand-Kirillov dimension if and only if the tuple $\mathfrak{g},\lambda,x$ occurs in Table 4.1.

Note that the braiding $c^f_{M,M}$ may depend on the choice of v, but the Gelfand-Kirillov dimension of $\mathcal{B}(M, c^f_{M,M})$ does not.
In Table 4.1 also the type of the extended Cartan matrix (b_{ij}) and the value $\varphi(\lambda,\lambda)$ are given. The weight λ_α always denotes the fundamental weight dual to the root α. The numbering of the roots is as in [13] and in Table 1.1.

Proof. Assume that the tuple $\mathfrak{g},\lambda,x$ leads to a Nichols algebra of finite Gelfand-Kirillov dimension and let $(a_{\alpha\beta})_{\alpha,\beta\in\Pi}$ be the Cartan matrix for $\mathfrak{g}$. Define for all $\alpha \in \Pi$ the integer $d_\alpha := \frac{(\alpha,\alpha)}{2}$. By Theorem 4.4.4 the extended Cartan matrix $(b_{ij})_{i,j\in P}$ is a finite Cartan matrix. Furthermore $P = \Pi\dot\cup\{\lambda\}$ and $b_{\alpha\beta} = a_{\alpha\beta}$ for $\alpha,\beta \in \Pi$. First assume that (b_{ij}) is not a connected Cartan matrix. As (a_{ij}) is a connected Cartan matrix observe

$$b_{\lambda\alpha} = 0 = b_{\alpha\lambda}\ \forall \alpha \in \Pi.$$

By the definition of (b_{ij}) this implies $\lambda = 0$. This is the first line in the table. Now assume that (b_{ij}) is a connected finite Cartan matrix and thus its Coxeter graph contains no cycles. As $(a_{\alpha\beta})$ is also a connected finite Cartan matrix there is a unique root $\alpha \in \Pi$ such that

$$b_{\alpha\lambda}, b_{\lambda\alpha} < 0 \text{ and for all } \beta \in \Pi \setminus \{\alpha\} : b_{\beta\lambda} = 0 = b_{\lambda\beta}.$$

Type of $\mathfrak{g}$	λ	x	Type of (b_{ij})	$\varphi(\lambda,\lambda)$	D
any	0	any	$D \cup A_0$	any	no relations
$A_n, n \geq 1$	λ_{α_1} or λ_{α_n}	$\frac{n+2}{n+1}$	A_{n+1}	2	2
$A_n, n \geq 1$	λ_{α_1} or λ_{α_n}	$\frac{1}{n+1}$	B_{n+1}	1	2
$A_n, n \geq 1$	$2\lambda_{\alpha_1}$ or $2\lambda_{\alpha_n}$	$\frac{4}{n+1}$	C_{n+1} (resp. B_2)	4	3
$A_n, n \geq 3$	$\lambda_{\alpha_{n-1}}$ or λ_{α_2}	$\frac{4}{n+1}$	D_{n+1}	2	2
A_1	λ_{α_1}	$\frac{1}{6}$	G_2	$\frac{3}{2}$	4
A_1	$3\lambda_{\alpha_1}$	$\frac{3}{2}$	G_2	6	2
A_5	λ_{α_3}	$\frac{1}{2}$	E_6	2	2
A_6	λ_{α_3} or λ_{α_4}	$\frac{2}{7}$	E_7	2	2
A_7	λ_{α_3} or λ_{α_5}	$\frac{1}{8}$	E_8	2	2
$B_n, n \geq 2$	λ_{α_1}	2	B_{n+1}	4	2
B_2	λ_{α_2}	1	C_3	2	2
B_3	λ_{α_3}	1	F_4	2	2
$C_n, n \geq 3$	λ_{α_1}	1	C_{n+1}	2	2
C_3	λ_{α_3}	1	F_4	4	2
$D_n, n \geq 5$	λ_{α_1}	1	D_{n+1}	2	2
D_4	$\lambda_{\alpha_1}, \lambda_{\alpha_3}$ or λ_{α_4}	1	D_5	2	2
D_5	λ_{α_4} or λ_{α_5}	$\frac{3}{4}$	E_6	2	2
D_6	λ_{α_5} or λ_{α_6}	$\frac{1}{2}$	E_7	2	2
D_7	λ_{α_6} or λ_{α_7}	$\frac{1}{4}$	E_8	2	2
E_6	λ_{α_1} or λ_{α_6}	$\frac{2}{3}$	E_7	2	2
E_7	λ_{α_7}	$\frac{1}{2}$	E_8	2	2

Table 4.1: Highest weights with Nichols algebras of finite Gelfand-Kirillov dimension; D is the degree of the relations calculated with Theorem 4.5.5

This implies that

$$l := (\alpha, \lambda) > 0 \text{ and for all } \beta \in \Pi \setminus \{\alpha\} : (\beta, \lambda) = 0.$$

Observe, using the definition of (b_{ij}) and φ, that

$$l = -\frac{b_{\alpha\lambda}(\alpha,\alpha)}{2} = -b_{\alpha\lambda}d_\alpha,$$

$$\varphi(\lambda,\lambda) = \frac{b_{\alpha\lambda}}{b_{\lambda\alpha}}(\alpha,\alpha) = 2\frac{b_{\alpha\lambda}}{b_{\lambda\alpha}}d_\alpha, \text{ and}$$
$$x = \varphi(\lambda,\lambda) - (\lambda,\lambda).$$

Furthermore we conclude $\lambda = -b_{\alpha\lambda}\lambda_\alpha$, where λ_α is the weight dual to the root α, i.e.

$$(\lambda_\alpha, \beta) = \delta_{\beta,\alpha}d_\alpha.$$

In a case-by-case analysis we will now consider all finite connected Cartan matrices $(a_{\alpha\beta})_{\alpha,\beta\in\Pi}$ and all possible finite connected Cartan matrices $(b_{ij})_{i,j\in\Pi\dot\cup\{\lambda\}}$ having $(a_{\alpha\beta})$ as a submatrix. In each case we compute the values for $l, \varphi(\lambda,\lambda)$ and x and decide if there is a tuple $\mathfrak{g}, \lambda, x$ leading to the matrix (b_{ij}). For every case also the Dynkin diagram of (b_{ij}) with labeled vertices is given. The vertices $1, \dots, n$ correspond to the simple roots $\alpha_1, \dots, \alpha_n \in \Pi$, the vertex $\star$ corresponds to $\lambda \in P$.
Note that to calculate x we must calculate the values $(\lambda_\alpha, \lambda_\alpha)$ for some $\alpha \in \Pi$. To do this we use Table 1 from [13, 11.4] and the explicit construction of the root systems there. However in the case of the root system B_n we have to multiply the scalar product by 2 to obtain the normalization described in Subsection 1.2.3.

$\underline{A_n \to A_{n+1}, n \geq 1}$:

$$\overset{1}{\bullet} — \overset{2}{\bullet}\cdots\cdots\overset{n-1}{\bullet} — \overset{n}{\bullet} — \overset{\star}{\bullet} \text{ or } \overset{\star}{\bullet} — \overset{1}{\bullet} — \overset{2}{\bullet}\cdots\cdots\overset{n-1}{\bullet} — \overset{n}{\bullet}$$

We have either $\alpha = \alpha_n$ or $\alpha = \alpha_1$. In any case $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$ and

$$\varphi(\lambda,\lambda) = 2, \lambda = \lambda_\alpha, (\lambda,\lambda) = \frac{n}{n+1}, x = \frac{n+2}{n+1}.$$

So $\lambda = \lambda_{\alpha_1}$ or $\lambda = \lambda_{\alpha_n}$ together with $x = \frac{n+2}{n+1}$ extend the matrix of A_n to A_{n+1}.

$\underline{A_n \to B_{n+1}, n \geq 1}$:

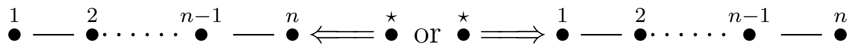
$$\overset{1}{\bullet} — \overset{2}{\bullet}\cdots\cdots\overset{n-1}{\bullet} — \overset{n}{\bullet} \Longrightarrow \overset{\star}{\bullet} \text{ or } \overset{\star}{\bullet} \Longleftarrow \overset{1}{\bullet} — \overset{2}{\bullet}\cdots\cdots\overset{n-1}{\bullet} — \overset{n}{\bullet}$$

Here either $\alpha = \alpha_1$ or $\alpha = \alpha_n$. In any case $d_\alpha = 1, b_{\alpha\lambda} = -1, b_{\lambda\alpha} = -2$ and

$$\varphi(\lambda, \lambda) = 1, \lambda = \lambda_\alpha, (\lambda, \lambda) = \frac{n}{n+1}, x = \frac{1}{n+1}.$$

So $\lambda = \lambda_{\alpha_1}$ or $\lambda = \lambda_{\alpha_n}$ together with $x = \frac{1}{n+1}$ extend the matrix of A_n to B_{n+1}.

<u>$A_n \to C_{n+1}, n \geq 1$</u>:

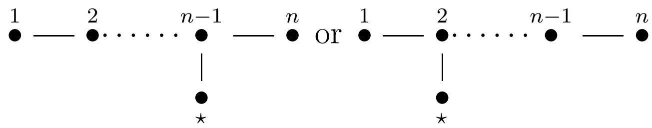


Either $\alpha = \alpha_1$ or $\alpha = \alpha_n$. In any case $d_\alpha = 1, b_{\alpha\lambda} = -2, b_{\lambda\alpha} = -1$ and

$$\varphi(\lambda, \lambda) = 4, \lambda = 2\lambda_\alpha, (\lambda, \lambda) = \frac{4n}{n+1}, x = \frac{4}{n+1}.$$

So $\lambda = 2\lambda_{\alpha_1}$ or $\lambda = 2\lambda_{\alpha_n}$ together with $x = \frac{4}{n+1}$ extend the matrix of A_n to C_{n+1} (resp. B_2).

<u>$A_n \to D_{n+1}, n \geq 3$</u>:

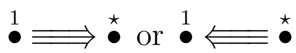


Here either $\alpha = \alpha_2$ or $\alpha = \alpha_{n-1}$. In any case $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$ and

$$\varphi(\lambda, \lambda) = 2, \lambda = \lambda_\alpha, (\lambda, \lambda) = \frac{2(n-1)}{n+1}, x = \frac{4}{n+1}.$$

So $\lambda = \lambda_{\alpha_2}$ or $\lambda = \lambda_{\alpha_{n-1}}$ together with $x = \frac{4}{n+1}$ extend the matrix of A_n to D_{n+1}.

<u>$A_1 \to G_2$</u>:



In any case $\alpha = \alpha_1$ and $d_\alpha = 1$. The left diagram means $b_{\alpha\lambda} = -1, b_{\lambda\alpha} = -3$ and

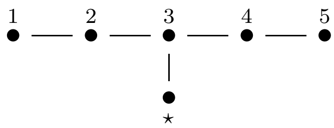
$$\varphi(\lambda, \lambda) = \frac{2}{3}, \lambda = -\lambda_{\alpha_1}, (\lambda, \lambda) = \frac{1}{2}, x = \frac{1}{6}.$$

The right diagram means $b_{\alpha\lambda} = -3, b_{\lambda\alpha} = -1$ and

$$\varphi(\lambda, \lambda) = 6, \lambda = 3\lambda_{\alpha_1}, (\lambda, \lambda) = \frac{9}{2}, x = \frac{3}{2}.$$

So $\lambda = \lambda_{\alpha_1}, x = \frac{1}{6}$ or $\lambda = 3\lambda_{\alpha_1}, x = \frac{3}{2}$ extend the matrix of A_1 to G_2.

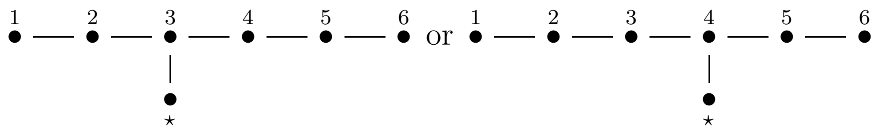
<u>$A_5 \to E_6$</u>:



Here $\alpha = \alpha_3$ and $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$. This implies

$$\varphi(\lambda, \lambda) = 2, \lambda = \lambda_{\alpha_3}, (\lambda, \lambda) = \frac{3}{2}, x = \frac{1}{2}.$$

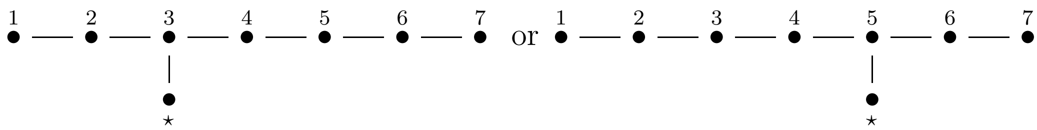
<u>$A_6 \to E_7$</u>:



$\alpha = \alpha_3$ or $\alpha = \alpha_4$. Furthermore $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$. This implies

$$\varphi(\lambda, \lambda) = 2, \lambda = \lambda_\alpha, (\lambda, \lambda) = \frac{12}{7}, x = \frac{2}{7}.$$

<u>$A_7 \to E_8$</u>:



$\alpha = \alpha_3$ or $\alpha = \alpha_5$. Furthermore $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$. This implies

$$\varphi(\lambda, \lambda) = 2, \lambda = \lambda_\alpha, (\lambda, \lambda) = \frac{15}{8}, x = \frac{1}{8}.$$

$\underline{B_n \to B_{n+1}, n \geq 2}$:

$$\overset{\star}{\bullet} \;—\; \overset{1}{\bullet} \;—\; \overset{2}{\bullet} \cdots\cdots \overset{n-1}{\bullet} \Longrightarrow \overset{n}{\bullet}$$

We have $\alpha = \alpha_1, d_\alpha = 2, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$ and this means

$$\varphi(\lambda, \lambda) = 4, \lambda = \lambda_\alpha, (\lambda, \lambda) = 2, x = 2.$$

$\underline{B_2 \to C_3}$:

$$\overset{1}{\bullet} \Longrightarrow \overset{2}{\bullet} \;—\; \overset{\star}{\bullet}$$

In this case $\alpha = \alpha_2$ and $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$. This means

$$\varphi(\lambda, \lambda) = 2, \lambda = \lambda_\alpha, (\lambda, \lambda) = 1, x = 1.$$

$\underline{B_3 \to F_4}$:

$$\overset{1}{\bullet} \;—\; \overset{2}{\bullet} \Longrightarrow \overset{3}{\bullet} \;—\; \overset{\star}{\bullet}$$

Here $\alpha = \alpha_3$ and $d_\alpha = 1, b_{\alpha\lambda} = b_{\lambda\alpha} = -1$. This means

$$\varphi(\lambda, \lambda) = 2, \lambda = \lambda_\alpha, (\lambda, \lambda) = 1, x = 1.$$

All the other cases follow the same idea and are omitted.
It remains to show that the data from the table lead to Nichols algebras of finite Gelfand-Kirillov dimension. It is clear that the braiding is of strong exponential type in every case. Moreover in each line the extended Cartan matrix (b_{ij}) is of finite type and thus the Nichols algebra has finite Gelfand-Kirillov dimension by Theorem 4.4.4. $\square$

4.5 Results on relations

In the preceding Section 4.4 we found out under which conditions the Nichols algebra of a $U_q(\mathfrak{g})$-module has finite Gelfand-Kirillov dimension. One of the next natural problems on these abstractly defined algebras is a representation by generators and relations. In this section we calculate $U_q^{\leq 0}(\mathfrak{g})$-module generators for the space of relations. Again we consider a more abstract setting first.
Let H be a Hopf algebra with bijective antipode, $V \in {}^H_H\mathcal{YD}$, $A := \mathcal{B}(V)\#H$ and M a graded Yetter-Drinfeld module over A of highest weight. Furthermore $T_c(M)$ resp. $T_c(M_H \oplus V)$ denote the tensor algebras of M resp. $M_H \oplus V$ viewed as braided Hopf algebras in the corresponding Yetter-Drinfeld categories ${}^A_A\mathcal{YD}$ resp. ${}^H_H\mathcal{YD}$. The following diagram of H-linear maps describes the situation of this section.

$$\begin{tikzcd}
T_c(M) \arrow[r, hook, shift left] \arrow[d, two heads, "\pi"'] & T_c(M)\#\mathcal{B}(V) \arrow[l, two heads, shift left, "T_c(M)\#\varepsilon"] \arrow[d, "\pi\#\mathcal{B}(V)"'] & T_c(M_H \oplus V) \arrow[l, two heads, "p"'] \arrow[d, two heads, "q"] \\
\mathcal{B}(M) \arrow[r, hook, shift left] & \mathcal{B}(M)\#\mathcal{B}(V) \arrow[l, two heads, shift left, "\mathcal{B}(M)\#\varepsilon"] \arrow[r, "\simeq"] & \mathcal{B}(M_H \oplus V)
\end{tikzcd}$$

The maps p and q are the unique algebra morphisms (and braided bialgebra morphisms) that restrict to the identity on $M_H \oplus V$. All maps but $T_c(M)\#\varepsilon$ and $\mathcal{B}(M)\#\varepsilon$ are algebra morphisms. The following proposition is the central tool of this section.

Proposition 4.5.1. Assume the situation described above. Fix a subset $X \subset T_c(M_H \oplus V)$ such that $H \cdot X$ generates $\ker q$ as an ideal. Furthermore write the elements of $p(X)$ in the form

$$p(x) = \sum_i m_i^x \# v_i^x \in T_c(M)\#\mathcal{B}(V)$$

with $m_i^x \in T_c(M), v_i^x \in \mathcal{B}(V)$. Consider the space

$$\hat{X} := \left\{ \sum_i m_i^x((v_i^x\#1) \cdot m) | x \in X, m \in T_c(M) \right\} \subset T_c(M).$$

Then $A \cdot \hat{X}$ generates $\ker \pi$ as an ideal.

Proof. Let $I := \ker q$. Obviously

$$\ker(\pi\#\mathcal{B}(V)) = p(I),$$

and it is easy to check that this implies

$$\ker\pi = (T_c(M)\#\varepsilon)p(I).$$

As $H \cdot X$ generates I as an ideal, $H \cdot p(X)$ generates $p(I)$ as an ideal. Now $T_c(M)\#\varepsilon$ is in general not an algebra morphism, so it is not easy to find ideal generators for $\ker\pi$. The elements of the form

$$\begin{aligned}(m'\#v)(h \cdot p(x))(m\#v') =& \\ = h_{(2)} \cdot \Big[&\Big(\mathbf{S}^{-1}(h_{(1)}) \cdot (m'\#v)\Big)\ p(x)\ \Big(\mathbf{S}(h_{(3)}) \cdot (m\#v')\Big)\Big]\end{aligned}$$

$(m, m' \in T_c(M), v, v' \in \mathcal{B}(V), x \in X, h \in H)$ generate $p(I)$ as a vector space. Thus $p(I)$ is generated as H-module by elements of the form

$$\begin{aligned}&(m'\#v)p(x)(m\#v') = \\ &= \sum_i m'\left((v^{(1)}\#v^{(2)}{}_{(-1)}) \cdot \left(m_i^x\left((v_i^{x(1)}\#v_i^{x(2)}{}_{(-1)}) \cdot m\right)\right)\right)\#v^{(2)}{}_{(0)}v_i^{x(2)}{}_{(0)}v'\end{aligned}$$

$(m, m' \in M, v, v' \in \mathcal{B}(V), x \in X)$. Now apply the H-linear map $T_c(M)\#\varepsilon$ and obtain H-module generators of $\ker\pi$ of the form

$$\sum_i m'\left((v\#1) \cdot \left(m_i^x\left((v_i^x\#1) \cdot m\right)\right)\right)$$

$(m, m' \in T_c(M), v \in \mathcal{B}(V), x \in X)$. Using that $T_c(M)$ is an H-module algebra conclude that elements of the form

$$m'\left((v\#h) \cdot \left(\sum_i m_i^x\left((v_i^x\#1) \cdot m\right)\right)\right)$$

$(m, m' \in T_c(M), v \in \mathcal{B}(V), h \in H, x \in X)$ generate $\ker\pi$ as vector space. This means that $A \cdot \hat{X}$ generates $\ker\pi$ as (left) ideal in $T_c(M)$. □

Remark 4.5.2. Assume that for all $x \in X$ there is $m^x \in T_c(M)$ such that

$$p(x) = m^x\#1.$$

Then $\hat{X}$ is the right ideal generated by the set $\{m^x | x \in X\}$. It is easy to check that in this case the $\mathcal{B}(V)\#H$-module generated by the $m^x, x \in X$ generates $\ker\pi$ as an ideal.

The quantum group case

The description of the generators of the ideal $\ker \pi$ obtained in the preceding theorem is not very explicit as the set $\hat{X}$ may be very large. Nevertheless it is sufficient for the case treated in Section 4.4 because then we are actually in the situation of Remark 4.5.2.

In this section we work over an algebraically closed field k of characteristic zero and assume that $q \in k$ is not a root of unity. Let M be a finite-dimensional integrable $U_q(\mathfrak{g})$-module with braiding c^f of strong exponential type with function φ; moreover assume that the extended Cartan matrix is a generalized Cartan matrix. Let $U = \mathcal{B}(V)\#kG$ be the extension defined in 3.4. We require that the ideal $\ker q$ is generated by the quantum Serre relations

$$\begin{aligned}
\forall \alpha, \beta \in \Pi, \alpha \neq \beta : \quad & r_{\alpha\beta} = \mathrm{ad}_c(\hat{F}_\alpha)^{1-b_{\alpha\beta}}(\hat{F}_\beta),\\
\forall \alpha \in \Pi, 1 \leq i \leq r : \quad & r_{i\alpha} = \mathrm{ad}_c(m_i)^{1-b_{i\alpha}}(\hat{F}_\alpha),\\
\forall \alpha \in \Pi, 1 \leq i \leq r : \quad & r_{\alpha i} = \mathrm{ad}_c(\hat{F}_\alpha)^{1-b_{\alpha i}}(m_i),\\
\forall 1 \leq i \neq j \leq r : \quad & r_{ij} = \mathrm{ad}_c(m_i)^{1-b_{ij}}(m_j).
\end{aligned}$$

Remark 4.5.3. Note that if φ is symmetric and (b_{ij}) is a symmetrizeable generalized Cartan matrix, then the braiding on $M_{kG} \oplus V$ is of Frobenius-Lusztig type by the proof of Theorem 4.4.4. In this case [4, Theorem 2.9] ensures that $\ker q$ is generated by the quantum Serre relations.

In order to apply Proposition 4.5.1 calculate the images of $r_{\alpha\beta}, r_{i\alpha}, r_{\alpha i}, r_{ij}$ under p. First observe

$$\begin{aligned}
p(r_{\alpha\beta}) &= \mathrm{ad}_c(p(\hat{F}_\alpha))^{1-b_{\alpha\beta}}(p(\hat{F}_\beta)) = \mathrm{ad}_c(1\#\hat{F}_\alpha)^{1-b_{\alpha\beta}}(1\#\hat{F}_\beta) =\\
&= 1\# \left(ad_c(\hat{F}_\alpha)^{1-b_{\alpha\beta}}(\hat{F}_\beta)\right) = 0
\end{aligned}$$

because this is a relation in $\mathcal{B}(V)$. For $r_{i\alpha}$ use the explicit form of the quantum Serre relations from [3, Equation A.8]:

$$\mathrm{ad}_c(x)^n(y) = \sum_{s=0}^{n} (-1)^s \binom{n}{s}_\gamma \gamma^{\frac{s(s-1)}{2}} \eta^s x^{n-s} y x^s,$$

if $c(x \otimes y) = \eta y \otimes x$ and $c(x \otimes x) = \gamma x \otimes x$. Define the coefficients $q_{xy}, x, y \in P$

as in the proof of Theorem 4.4.4. Then

$$\begin{aligned}
p(r_{i\alpha}) &= p\left(\sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s m_i^{1-b_{i\alpha}-s}\hat{F}_\alpha m_i^s\right)\\
&= \sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s (m_i^{1-b_{i\alpha}-s}\#1)(1\#\hat{F}_\alpha)(m_i^s\#1)\\
&= \sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s \left(m_i^{1-b_{i\alpha}-s}(K_\alpha\cdot m_i^s)\#\hat{F}_\alpha\right)\\
&\quad+\sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s \left(m_i^{1-b_{i\alpha}-s}(\hat{F}_\alpha\cdot m_i^s)\#1\right)
\end{aligned}$$

The first summand is zero. This can be seen using

$$q_{i\alpha}q_{\alpha i} = q_{ii}^{b_{i\alpha}}$$

and [3, Equation A.5]:

$$\begin{aligned}
&\sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s \left(m_i^{1-b_{i\alpha}-s}(K_\alpha\cdot m_i^s)\#\hat{F}_\alpha\right)\\
=&\sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s q_{\alpha i}^s \left(m_i^{1-b_{i\alpha}}\#\hat{F}_\alpha\right)\\
=&\left(\sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s+1)}{2}} q_{ii}^{-s(1-b_{i\alpha})}\right)\left(m_i^{1-b_{i\alpha}}\#\hat{F}_\alpha\right) = 0.
\end{aligned}$$

Thus the image of $r_{i\alpha}$ is

$$p(r_{i\alpha}) = \left(\sum_{s=0}^{1-b_{i\alpha}}(-1)^s\binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{i\alpha}^s m_i^{1-b_{i\alpha}-s}(\hat{F}_\alpha\cdot m_i^s)\right)\#1.$$

$r_{\alpha i}$ is mapped to

$$\begin{aligned}
p(r_{\alpha i}) &= \mathrm{ad}_c(p(\hat{F}_\alpha))^{1-b_{\alpha i}}(p(m_i)) = \mathrm{ad}_c(1\#\hat{F}_\alpha)^{1+\frac{2(\alpha,\lambda_i)}{(\alpha,\alpha)}}(m_i\#1) =\\
&= \left(\hat{F}_\alpha^{1+\frac{2(\alpha,\lambda_i)}{(\alpha,\alpha)}}\cdot m_i\right)\#1 = 0
\end{aligned}$$

because the braided adjoint action of $\mathcal{B}(V)$ on M (in $T_c(M)\#\mathcal{B}(V)$) is the same as the module action (denoted by $\cdot$) of $\mathcal{B}(V)\subset U_q(\mathfrak{g})$ on M. By [15, 5.4.] the last equality holds. This leaves r_{ij} to be considered.

$$\begin{aligned} p(r_{ij}) &= p\left(\mathrm{ad}_c(m_i)^{1-b_{ij}}(m_j)\right) = \mathrm{ad}_c\left(p(m_i)\right)^{1-b_{ij}}\left(p(m_j)\right) \\ &= \mathrm{ad}_c(m_i\#1)^{1-b_{ij}}(m_j\#1) = \mathrm{ad}_c(m_i)^{1-b_{ij}}(m_j)\#1 \end{aligned}$$

Remark 4.5.4. A short calculation results in the following representation of the relation coming from $r_{i\alpha}$ for $1 \le i \le r, \alpha \in \Pi$:

$$R_{i\alpha} := \sum_{t=0}^{-b_{i\alpha}} \left[\sum_{s=t+1}^{1-b_{i\alpha}} (-1)^s \binom{1-b_{i\alpha}}{s}_{q_{ii}} q_{ii}^{\frac{s(s-1)}{2}} q_{ii}^{sb_{i\alpha}} \right] q_{\alpha i}^{-t} m_i^{-b_{i\alpha}-t} \left(F_\alpha \cdot m_i\right) m_i^t.$$

The relations coming from the $r_{ij}, 1 \le i, j \le r$ are

$$R_{ij} := \mathrm{ad}_c(m_i)^{1-b_{ij}}(m_j)$$

It follows from Proposition 4.5.1 that the $\mathcal{B}(V)\#kG$ submodule of $T_c(M)$ generated by the elements

$$\{R_{i\alpha} | 1 \le i \le r, \alpha \in \Pi\} \cup \{R_{ij} | 1 \le i \ne j \le r\}$$

generates the kernel of the canonical map

$$\pi : T_c(M) \to \mathcal{B}(M)$$

as an ideal.

Theorem 4.5.5. Let M be a finite-dimensional integrable $U_q(\mathfrak{g})$-module and fix a braiding c^f of strong exponential type with symmetric function φ. Assume that the extended Cartan matrix $(b_{ij})_{i,j\in P}$ is a generalized Cartan matrix. Consider the grading on $\mathcal{B}(M)$ such that the elements of M have degree 1. Then $\mathcal{B}(M)$ is generated by M with homogeneous relations of the degrees

$$2 - b_{ij} \quad \text{for} \quad 1 \le i \ne j \le r \text{ and}$$
$$1 - b_{i\alpha} \quad \text{for} \quad 1 \le i \le r, \alpha \in \Pi \text{ such that } b_{i\alpha} \ne 0.$$

The last column of Table 4.1 was calculated using this theorem.

Proof. Exactly as in the proof of Theorem 4.4.4 we see that $(b_{ij})_{i,j\in P}$ is a symmetric generalized Cartan matrix. By Remark 4.5.3 the Nichols algebra of $M_{kG} \oplus V$ is given by the quantum Serre relations.
Realize the module M as a Yetter-Drinfeld module over $\mathcal{B}(V)\#kG$ as in Section 3.4. The $\mathcal{B}(V)\#kG$-module generated by the elements $R_{i\alpha}, R_{ij}$ with $1 \le i \ne j \le r, \alpha \in \Pi$ generates the $\ker \pi$ as an ideal. $R_{i\alpha}$ has degree $1 - b_{i\alpha}$,

R_{ij} has degree $2 - b_{ij}$ (with respect to the grading of $T_c(M)$ giving M the degree 1). As the homogeneous components of $T_c(M)$ are $\mathcal{B}(V)\#kG$-modules all defining relations can be found in the degrees

$$1 - b_{i\alpha} \text{ and } 2 - b_{ij}.$$

Observe that $p(r_{i\alpha})$ is zero if $b_{i\alpha} = 0$: The summand for $s = 0$ is zero anyway because $\varepsilon(\hat{F}_\alpha) = 0$. The summand for $s = 1$ is a scalar multiple of $\hat{F}_\alpha \cdot m_i$. If $b_{i\alpha} = 0$ then also $b_{\alpha i} = 0$ and thus by [15, 5.4.] $\hat{F}_\alpha \cdot m_i = 0$. □

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Summary

This thesis deals with the structure of braided Hopf algebras of triangular type. Braided Hopf algebras arise naturally in the structure theory of usual Hopf algebras. A braided Hopf algebra is of triangular type, if it is generated by a finite-dimensional braided subspace of primitive elements and if moreover the braiding on this subspace is triangular. Nichols algebras of $U_q(\mathfrak{g})$-modules are important examples.

One of the main results of this thesis is Theorem 2.2.4, which shows the existence of bases of Poincaré-Birkhoff-Witt (PBW) type for braided Hopf algebras of triangular type. The PBW-basis is described by Lyndon words in the generators of the algebra. The combinatorial proof basically follows a paper of Kharchenko, where he proves a PBW-result for so-called character Hopf algebras, but our situation requires new methods and ideas.

As one application of our PBW-theorem we prove a PBW-result for Hopf algebras which are generated by an abelian group and a finite-dimensional G-subspace of skew-primitive elements. This generalizes the original result of Kharchenko in the sense that the action of the group on the skew-primitive elements is not necessarily given by a character.

As a second application we use the PBW-theorem to determine the structure of Nichols algebras of low-dimensional $U_q(\mathfrak{sl}_2)$-modules, where the braiding is given by the quasi-$\mathcal{R}$-matrix.

The second main result in Chapter 3 of this thesis gives a characterization of triangular braidings. Originally these braidings are defined by a certain combinatorial property. We show that triangular braidings are exactly those braidings coming from Yetter-Drinfeld modules over pointed Hopf algebras with abelian coradical which are completely reducible as modules over the coradical. Braidings induced by the quasi-$\mathcal{R}$-matrix on $U_q(\mathfrak{g})$-modules are triangular. We show how they arise in this context.

Answering a question of Andruskiewitsch [1], we investigate the structure of Nichols algebras of $U_q(\mathfrak{g})$-modules in Chapter 4 of this thesis. We describe a method that allows to reduce the study of these Nichols algebras to the study of Nichols algebras with diagonal braiding. We apply this method to decide when the Gelfand-Kirillov dimension of these algebras is finite and to describe their defining relations. We give a complete list of all simple $U_q(\mathfrak{g})$-modules ($\mathfrak{g}$ a finite-dimensional simple complex Lie algebra), that have Nichols algebras with finite Gelfand-Kirillov dimension.

Zusammenfassung

Diese Dissertation beschäftigt sich mit der Struktur verzopfter Hopfalgebren vom triangulären Typ. Verzopfte Hopfalgebren treten in natürlicher Weise in der Strukturtheorie üblicher Hopfalgebren auf. Eine verzopfte Hopfalgebra ist vom triangulären Typ, falls sie von einem endlichdimensionalen verzopften Unterraum primitiver Elemente erzeugt wird und die Verzopfung auf diesem Unterraum triangulär ist. Wichtige Beispiele sind Nicholsalgebren von $\mathcal{R}$-Matrix-Verzopfungen auf $U_q(\mathfrak{g})$-Moduln.
Eines der Hauptresultate dieser Arbeit ist Theorem 2.2.4, das die Existenz einer Basis vom Poincaré-Birkhoff-Witt (PBW) Typ für verzopfte Hopfalgebren vom triangulären Typ zeigt. Die PBW-Basis wird durch Lyndonwörter in den Erzeugenden der Algebra beschrieben. Der kombinatorische Beweis orientiert sich an einer Arbeit Kharchenkos, in der ein PBW-Resultat für sogenannte Charakterhopfalgebren bewiesen wird. Allerdings erfordert unsere Situation einige neue Methoden und Ideen.
Als eine Anwendung des PBW-Satzes beweisen wir ein PBW-Resultat für Hopfalgebren, die von einer abelschen Gruppe G und einem endlichdimensionalen G-Unterraum von schiefprimitiven Elementen erzeugt werden. Dies verallgemeinert das Resultat Kharchenkos in dem Sinne, dass die Wirkung der Gruppe auf den schiefprimitiven Elementen nicht mehr durch einen Charakter der Gruppe gegeben sein muss.
Als eine zweite Anwendung benutzen wir den PBW-Satz, um die Struktur der Nicholsalgebren niedrigdimensionaler $U_q(\mathfrak{sl}_2)$-Moduln zu bestimmen, wobei die Verzopfung durch die quasi-$\mathcal{R}$-Matrix gegeben ist.

Als zweites Hauptergebnis wird in Kapitel 3 eine Charakterisierung triangulärer Verzopfungen gegeben. Diese sind ursprünglich durch eine kombinatorische Bedingung definiert. Hier zeigen wir, dass trianguläre Verzopfungen genau diejenigen Verzopfungen sind, die von Yetter-Drinfeld-Moduln über punktierten Hopfalgebren mit abelschem Koradikal induziert werden, welche als Moduln über dem Koradikal halbeinfach sind. Verzopfungen, die durch die quasi-$\mathcal{R}$-Matrix auf $U_q(\mathfrak{g})$-Moduln induziert werden, sind triangulär. Wir zeigen, wie sie in dieses Bild passen.

Im Kapitel 4 der Arbeit geht es, motiviert durch eine Frage von Andruskiewitsch [1], um die Struktur der Nicholsalgebren von $U_q(\mathfrak{g})$-Moduln. Wir beschreiben eine Methode, mit der man die Untersuchung dieser Nicholsalgebren auf die Theorie von Nicholsalgebren mit diagonaler Verzopfung zurückführen kann. Wir wenden diese Methode an, um ein Kriterium für die Endlichkeit der Gelfand-Kirillov-Dimension dieser Algebren zu beweisen und um ihre definierenden Relationen zu beschreiben. Wir geben eine vollständige

Liste aller einfachen $U_q(\mathfrak{g})$-Moduln ($\mathfrak{g}$ eine endlichdimensionale einfache komplexe Liealgebra) an, deren Nicholsalgebren endliche Gelfand-Kirillov-Dimension haben.

Lebenslauf

Persönliche Daten

Stefan Ufer
geboren am 4. Oktober 1975 in Garmisch-Partenkirchen

Schulausbildung

1982-1986 Besuch der Grundschule Murnau am Staffelsee

1986-1995 Besuch des Staffelsee-Gymnasiums in Murnau am Staffelsee, Abschluss mit Abitur

Zivildienst

Juli 1995 bis Juli 1996 Zivildienst in der evangelischen Kirchengemeinde Murnau am Staffelsee

Hochschulbildung

November 1996 Beginn des Studiums für Lehramt an Gymnasien mit Fächerkombination Mathematik, Physik and der LMU München

April und Oktober 1998 Staatliche Zwischenprüfungen

April 1999 Vordiplom in Mathematik (Nebenfach Experimentalphysik)

Mai 1999 bis Mai 2002 Mitglied der Studienstiftung des deutschen Volkes

SoSe 2000 Staatsexamen im erziehungswissenschaftlichen Bereich

November 2001 Abschluss des Mathematikstudiums mit Diplom

SoSe 2002 Abschluss des Lehramtsstudiums mit Staatsexamen

Seit September 2002 Stipendiat der Graduiertenförderung des bayerischen Staates

Berufstätigkeit

Mai 2002 bis August 2002 wissenschaftlicher Mitarbeiter am mathematischen Institut der LMU München