WiSe 2018/19

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Symplectic geometry

Exercise sheet 9

Exercise 1. Let (M, ω) be a compact symplectic manifold and

$$P_\omega = \left\{ \int_\Sigma \omega \,\middle|\, [\Sigma] \in H_2(M; \mathbb{Z}) \right\}.$$

Show that for all $\gamma \in H_1(M; \mathbb{Z})$

$$\left\{ \int_\gamma \mathrm{Flux}(\phi_t) \,\middle|\, [\phi_t] \in \pi_1(\mathrm{Symp}_0(M, \omega), \mathrm{id}) \right\} \subset P_\omega.$$

Conclude that if Γ_ω is discrete when ω represents a rational (singular cohomology class). Recall that the de Rham isomorphism establishes $H^*_{dR}(M; \mathbb{R}) \simeq H^*(M; \mathbb{R})$ and $H^*(M; \mathbb{Q})$ (containing the rational classes) can be viewed as a subset of $H^*(M; \mathbb{R})$.

Exercise 2. Let (M^{2n}, ω) be a compact symplectic manifold and $\Omega = \omega^n$ the associated volume form. Since every symplectic map preserves Ω there is a natural inclusion $\iota : \mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M, \Omega)$ into the group of volume preserving diffeomorphisms.

a) Define a map

$$\mathrm{VolFlux} : \widetilde{\mathrm{Diff}}_0(M, \Omega) \longrightarrow H^{2n-1}_{dR}(M; \mathbb{R})$$

analogous to the Flux homomorphism and verify that it is a well defined homomorphism. (You may assume that $\mathrm{Diff}_0(M, \Omega)$ is sufficiently connected so that the universal cover is well defined.)

b) Let $L_{\omega,1} : H^1(M, \mathbb{R}) \longrightarrow H^{2n-1}(M; \mathbb{R})$ be the multiplication with $n\omega^{n-1}$. Prove that

$$L_{\omega,1} \circ \mathrm{Flux} = \mathrm{VolFlux} \circ \iota.$$

Exercise 3. Let $F \subset (M, \omega)$ be a closed submanifold and $\phi_t : U \longrightarrow M$ a symplectic isotopy defined on a tubular neighborhood U of F so that ϕ_0 is the inclusion $U \hookrightarrow M$. We are interested in the existence of a symplectic isotopy ψ_t of M so that $\psi_t|_F = \phi_t|_F$ for all t.

a) Let $\Sigma \subset M$ be an oriented compact surface with boundary $\partial\Sigma \subset F$ and $\gamma = \partial\Sigma$. Interpret the expression on the left hand side of the condition

$$\int_{\partial\Sigma} \mathrm{Flux}([\phi_t]) = 0 \tag{1}$$

and show that if it is not satisfied, then there is no symplectic isotopy ψ_t of M so that $\psi_t = \phi_t$on F for all t.

b) Give an example where there is no symplectic isotopy of M which coincides with ϕ_t on F.

c) Assume that ϕ_t is given by a Hamiltonian flow. Show that ψ_t exists and can be chosen to be Hamiltonian.

- Remark: (1) can be interpreted that $\partial_* H_2(M, F; \mathbb{R})$ lies in the kernel of the deRham cochain $\text{Flux}([\phi_t])$. It turns out that if (1) is satisfied for all $[\Sigma] \in H_2(M, F; \mathbb{Z})$, then the extension ψ_t of $\phi_t|_F$ exists.

Exercise 4. Let $\phi_t, t \in \mathbb{R}$ be a smooth flow of a vector field V on the manifold M.

a) Give an example where ϕ_t has a non-isolated invariant set.

b) Let $p \in M$ so that $\dot{\phi}_t(p) \neq 0$ and $\phi_T(p)$ for $T > 0$ be a closed orbit of ϕ. Fix a hypersurface Σ through p transverse to $\dot{\phi}_t(p)$ and consider that $\text{pr} \circ D\Phi_T : T_p\Sigma \longrightarrow T_p\Sigma$ where $\text{pr} : T_pM \longrightarrow T_p\Sigma$ is the projection along Σ.

   Is it true that γ is an isolated invariant set if and only if $\text{pr} \circ D\Phi_T$ does not have one as an eigenvalue? What if no complex zeroes of the characteristic polynomial of $\text{pr} \circ D\Phi_T$ lie on S^1.

c) Consider the vector field $V = r^2\partial_\varphi + (1-r)r\partial r$ on $\mathbb{R}^2$ in terms of polar coordinates. Find all invariant sets and the corresponding index pairs. Determine the homotopy type of the corresponding index space.

Hand in on Wednesday December 19 during the exercise class.