WiSe 2018/19

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# Symplectic geometry

## Exercise sheet 8

**Exercise 1.** Let $\omega = d\lambda$ be an exact symplectic form on a manifold and $\phi$ a compactly support symplectomorphism which is isotopic to the identity in $\mathrm{Symp}_0^c(M,\omega)$.

a) Show that $\phi^*\lambda - \lambda$ represents a class $F(\phi) \in H^1_{cpt}(M)$ (compactly supported de Rham cohomology).

b) Prove that $F : \mathrm{Symp}_0^c(M,\omega) \longrightarrow H^1_{cpt}(M)$ is a group homomorphism.

c) Show that $F(\phi) = 0$ for $\phi \in \mathrm{Ham}^c(M,\omega)$.

**Exercise 2.** The definition in exercise 1 works in the same way for $\phi \in \mathrm{Symp}_0(M,\omega)$. Consider the case $(M = T^*N, \omega = d\lambda_{st})$ in this setting and for a closed 1-form $\alpha \in \Omega^1(N)$ define

$$\begin{aligned} \phi_\alpha : T^*N &\longrightarrow T^*N \\ \gamma \in T^*_n N &\longrightarrow \gamma + \alpha(n). \end{aligned}$$

a) Prove that $\phi_\alpha$ is a symplectomorphism.

b) Determine $F(\phi_\alpha)$.

**Exercise 3.** Assume that $(M, d\lambda)$ is symplectic. We consider $\mathrm{Symp}_0^c(M,\omega)$ with the direct limit $C^1$-topology. Out goal is to show that $\mathrm{Ham}^c(M,\omega)$ is not simple. For this we define a homomorphism

$$\begin{aligned} \mathrm{Cal} : \mathrm{Ham}^c(M,\omega) &\longrightarrow \mathbb{R} \\ \phi &\longmapsto \int_M F\omega^n. \end{aligned}$$

where $\phi^*\lambda - \lambda = dF$ and $F$ has compact support.

a) Show that Cal is a homomorphism.

b) Assume that $H$ is compactly supported and let $\phi$ be the Hamiltonian time 1-map associated to $H$. Compute $\mathrm{Cal}(\phi)$. This shows that $\mathrm{Ham}^c(M,\omega)$ is not simple.

**Exercise 4.** Assume that $\lambda$ and $\lambda'$ are two primitives of $\omega$. Show that defining Cal using $\lambda$ or $\lambda'$ gives the same result. Use that $\lambda - \lambda' = \alpha$ is closed and $\phi^*\alpha - \alpha \in H^1_{cpt}$ is exact because $\phi^*$ is homotopic to the identity through compactly supported diffemorphisms.

Hand in on Wednesday December 12 during the exercise class.