WiSe 2018/19

Prof. Dr. Thomas Vogel
Daniel Räde

Symplectic geometry

Exercise sheet 7

Exercise 1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be smooth. f is strictly convex if $f''(x) > 0$ for all $x \in \mathbb{R}$. Show that for strictly convex functions the following are equivalent:

1. f has a critical point.
2. f has a local minimum.
3. f has a unique global minimum.
4. $\lim_{x\to\pm\infty} f(x) = \infty$.

Strictly convex functions satisfying any of these conditions are *stable*. Show that $f_a(x) = e^x + ax$ is stable if and only if $a > 0$.

Exercise 2. Let $F : V \longrightarrow \mathbb{R}$ be a smooth function on a finite dimensional vector space V. It is called strictly convex if its restriction to every affine line in V is strictly convex.

a) Show that F is strictly convex if and only if the quadratic form

$$\begin{aligned} d_p^2 F : V &\longrightarrow \mathbb{R} \\ v &\longmapsto \left.\frac{d^2}{dt^2}\right|_{t=0} F(p + tv) \end{aligned}$$

is positive definite for all p.

b) Generalize exercise 1 to the present case. As before, strictly convex functions satisfying any of the conditions (1),...,(4) are called stable.

c) Recall that for all $p \in V$ there is a canonical identification $T_p^* V \simeq V^*$. Thus one can define the *Legendre transformation*

$$\begin{aligned} L_F : V &\longrightarrow V^* \\ p &\longmapsto dF_p. \end{aligned}$$

Prove that this is a local diffeomorphism everywhere if F is strictly convex.

Exercise 3. Let $F : V \longrightarrow \mathbb{R}$ be strictly convex and

$$S_F := \left\{ \lambda \in V^* \;\middle|\; \begin{aligned} F_\lambda : V &\longrightarrow \mathbb{R} \\ p &\longmapsto F(p) - \lambda(p) \end{aligned} \quad \text{is stable} \right\}$$

a) Show that S_F is convex, open and that $L_F : V \longrightarrow S_F$ is a diffemorphism. Moreover, if $\lambda \in S_F$, then $L_F^{-1}(\lambda)$ is the unique minimum point of F_λ.

b) Assume that there is a positive definite quadratic form $Q(x)$ and $K \in \mathbb{R}$ so that $F(p) \geq Q(p) - K$ (such functions have *at least quadratic growth at infinity*). Prove that $S_V = V^*$ so that $L_F : V \longrightarrow V^*$ is a diffeomorphism.

Exercise 4. For a strictly convex function $F : V \longrightarrow \mathbb{R}$ consider $\widehat{F} : S_F \longrightarrow \mathbb{R}$ with $\widehat{F}(\lambda) = -\min_{p \in V}(F_\lambda(p))$. Prove that for all $p \in V$ and $\lambda \in S_F$

$$F(p) + \widehat{F}(\lambda) \geq \lambda(p).$$

Exercise 5. Let V be a finite dimensional vector space and α the canonical 1-form on $V \times V^* \simeq T^*V$, i.e. $\alpha((v,\beta)) = \lambda(v)$ for $(v,\beta) \in T_{(w,\lambda)}(V \times V^*)$. The form $\widehat{\alpha}$ is the canonical 1-form on $(V \times V^*)^* = V^* \times V^{**} \simeq V \times V^*$. Thus α and $\widehat{\alpha}$ can be viewed as forms on the same vector space.

Let Λ_F be the graph of L_F in $V \times V^*$. This is a Lagrangian submanifold for both ω and $\widehat{\omega}$ and $i^* : \Lambda_F \longrightarrow V \times V^*$ denotes the inclusion.

a) Prove that under this identification

$$\alpha + \widehat{\alpha} = d\gamma$$

where $\gamma : V \times V^* \longrightarrow \mathbb{R}$ is the function $\gamma(v,\lambda) = \lambda(v)$. Conclude that the symplectic forms $\omega = d\alpha$ and $\widehat{\omega} = d\widehat{\alpha}$ satisfy $\omega = -\widehat{\omega}$.

Prove that $\Lambda_{\widehat{F}}$ coincides with Λ_F under the identification $V \times V^* \simeq V^* \times V^{**}$.

b) Let $F : V \longrightarrow \mathbb{R}$ be strictly convex and assume that F has at least quadratic growth at infinity. According to exercise 3b this implies $S_F = V^*$. Prove that $i^*\alpha = \mathrm{pr}_1^* dF$ where $\mathrm{pr}_1 : \Lambda_F \longrightarrow V$ denotes the projection on the first factor of $V \times V^*$ (pr_2 is the projection on the second factor). Argue why

$$i^*\widehat{\alpha} = \mathrm{pr}_2^*(d\widehat{F}) = d\left(i^*\gamma - \mathrm{pr}_2^* F\right)$$

and conclude that $\widehat{\widehat{F}} - F$ is a constant ($\widehat{L}$ is the Legendre transformation for strictly convex functions with domain V^*).

Hand in on Wednesday November, 5 during the exercise class.