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Lecture in the winter term 2019/20

Contact topology

Please note: These notes summarize the content of the lecture. Many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. We will not attempt to give the first reference where a theorem appeared. Some proofs might take two lectures although they appear in a single lecture in these notes. Changes to this script are made without further notice at unpredictable times. If you find any typos or errors, please let us know.

1. Lecture on October 16 – Definition, examples, Gray

- We will be working in the smooth category unless noted otherwise.
- **Definition:** A *contact structure* on a manifold M of dimension $2n+1$ is a smooth hyperplane field ξ such that every point has a neighborhood U so that $\ker(\alpha) = \xi|_U$ so that

$$\alpha \wedge d\alpha^n \neq 0. \tag{1}$$

- **Remark:** If α' is another with $\ker(\alpha') = \xi$, then the condition (1) also holds for α'. (1) can be rephrased as follows: $d\alpha$ is a non-degenerate 2-form on $\ker(\alpha)$.

  For all n this condition can be reformulated in terms of commutators of local sections of ξ. When $n = 1$ this is very transparent: ξ is a contact structure if and only if $[X, Y] \neq \xi$ whenever X, Y are linearly independent local sections of ξ.
- **Reminder:** If ω is a non-degenerate 2-form on a vector space, then the vector space has even dimension $2n$ and there is a symplectic basis X_i, W_i, i.e.

$$\omega(X_i, X_j) = 0 \qquad \omega(Y_i, Y_j) = 0 \qquad \omega(X_i, Y_j) = \delta_{ij}.$$

  If ω is a two form on a vector space V of odd dimension, then the kernel of the map

$$\begin{aligned} V &\longrightarrow V* \\ X &\longmapsto (Y \longmapsto \omega(X, Y)) \end{aligned}$$

  is not empty, the image of this map has always even dimension.
- **Example -1:** If M has dimension 1, then M is has unique contact structure
- **Example 0:** $M = \mathbb{R}^{2n+1}$ and $\xi = \ker(\alpha)$ with $\alpha = dz - \sum_{i=1}^n y_i dx_i$.
- **Example 1:** $M = S^{2n+1} \subset \mathbb{C}^{n+1}$ and $\xi = TS^{2n+1} \cap iTS^{2n+1}$, i.e. $\xi(p)$ is the set of hyperplanes in T_pS^{2n+1} which are complex subspaces in $T_p\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$. A 1-form defining ξ is

$$\alpha(x_0, y_0, \ldots, x_n, y_n) = \frac{1}{2}\sum_{i=0}^n (x_i dy_i - y_i dx_i).$$

- **Example 2:** Let $(X, \omega = d\lambda)$ an exact symplectic manifold, i.e. ω is an exact, non-degenerate symplectic form. Then $\mathbb{R} \times X$ has a contact structure defined by $dt + \lambda$.

- **Example 3:** There is an important class of flows which are called Anosov flows: Let M be a 3-manifold and X a vector field such that the corresponding flow φ_t is Anosov. This means that there is a φ-invariant splitting $TM = \mathbb{R}X \oplus \zeta^{st} \oplus \zeta^{un}$ together with a Riemannian metric g on M and positive constants C, λ so that
$$\|D\varphi_t(X_{st})\| \leq Ce^{-\lambda t}\|X_{st}\| \qquad \|D\varphi_t(X_{un})\| \geq C^{-1}e^{\lambda t}\|X_{un}\|.$$
We assume that ζ^{st}, ζ^{un} are orientable and pick unit vector fields X_{st}, X_{un}. If $\zeta^{st}\zeta^{un}$ are *smooth* (this is a serious assumption which we make to simplify the exposition), hen $\xi_\pm = \mathrm{span}\{X, X_{st} \pm X_{un}\}$ is a pair of contact structures such that the induced orientation on M (with $n = 1$) are different. This situation appears for example, when M is the unit tangent bundle of tangent vectors on a hyperbolic surface and X defines the geodesic flow.
- **Example 4:** Let N be a smooth manifold and
$$\mathrm{pr} : M = (TN^* \setminus N = \{\text{zero section}\})/ \ \longrightarrow N$$
where $\alpha \sim \alpha'$ if and only if $\alpha = \lambda\alpha'$ for $\lambda \in \mathbb{R}$. Then the tautological distribution ξ with
$$\xi([\alpha]) = \{X \in T_{[\alpha]}M \mid \alpha(\mathrm{pr}_*(X)) = 0\}$$
is a contact structure.
- **Convention:** We will usually assume that a contact structure is coorienable, i.e. there is a *global* 1-form α, so called *contact forms*, whose kernel is ξ.
- **Definition:** Let α be a contact form on M. Then there is a unique vector field R_α so that $\alpha(R_\alpha) \equiv 1$ and $d\alpha(R_\alpha, \cdot) \equiv 0$. Then R_α is the *Reeb vector field.*
- **Fact:** The contact condition on a hyperplane field is open in the (fine) C^1-topology.
- **Definition:** Let (M, ξ) and (M', ξ') be two contact manifolds of equal dimension. A *contact diffeomorphism/contact transformation* $f : M \longrightarrow M'$ is a diffeomorphism such that $Df(\xi) \subset \xi'$. Contact transformations preserving fixed contact forms are sometimes called *strict.*
- **Remark:** Since $L_{R_\alpha}\alpha = 0$, the flow of R_α consists of strict contact transformations.
- **Example:** Let $M = \mathbb{R}^3$ and $\alpha_0 = dz - ydx$. Using polar coordinates one defines $\alpha_1 = dz + \rho^2 d\vartheta$. The diffeomorphism
$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x, 2y, (z + xy)) \end{aligned}$$
satisfies $f^*\alpha_0 = \alpha_1$ since
$$\begin{aligned} \alpha_1 = dz + \rho^2 d\vartheta &= dz + xdy - ydx \\ &= d(z + xy) - (2y)dx. \end{aligned}$$
In this example, even the contact forms are isomorphic. In particular, Df maps the Reeb vector field of α_1 to the Reeb vector fields of α_0.
- **Example:** There is no strict contact transformation between the contact forms (polar coordinates on $\mathbb{R}^3$)
$$\begin{aligned} \alpha_0 &= dz + \rho^2 d\vartheta \\ \alpha_1 &= \frac{1}{(1 + z^2 + \rho^2)^2}\alpha_0 \end{aligned}$$

for the same contact structure on $\mathbb{R}^3$. This can be seen using properties of the Reeb vector field: The Reeb vectorfield R_0 of α_0 is ∂_z, so no orbit is closed. Tangent vectors to curves with

$$\dot{\vartheta} = 1 \qquad \dot{z} = \frac{1 + z^2 - \rho^2}{2} \qquad \dot{\rho} = \rho z$$

are contained in the kernel of $d\alpha_1$. In particular, the circle $\{z = 0, \rho = 1\}$ is a closed flow line of the Reeb vector field of α_1.

- **Example:** In the situation of example 4, let $f : N \longrightarrow N$ be a diffeomorphism. Then

$$\begin{aligned} F : M &\longrightarrow M \\ [\alpha] &\longmapsto [f^{-1*}\alpha]. \end{aligned}$$

transformation

- The following Theorem is an application of the so-called Moser method. This is first used in [6] to prove that two volume forms with the same sign and volume are diffeomorphic.
- **Theorem(Gray):** Let $\alpha_t, t \in [0,1]$, be a smooth family of contact structures on M such that ξ_t is constant outside of a compact set (in the interior of M if M has boundary). Then there is a 1-parameter family of contact transformations φ_t, i.e. there is a family of functions f_t on M such that

$$\varphi_t^* \alpha_t = f_t \alpha_0.$$

- **Proof:** The key idea is to construct a time dependent vector field whose flow φ_t has the desired property. Let X_t be a smooth, compactly supported vector field with flow φ_t. Then according to the chain rule

$$\frac{d}{dt}\varphi_t^* \alpha_t = \varphi_t^* \left(\frac{d\alpha_t}{dt} + i_{X_t} d\alpha_t + d(i_{X_t}\alpha_t) \right).$$

We seek X_t so that there are positive functions f_t such that $\varphi_t \alpha_t = f_t \alpha_0$. This means

$$\begin{aligned} \frac{df_t}{dt}\alpha_0 &= \frac{d\log(f_t)}{dt}\varphi_t^*\alpha_t \\ &= \varphi_t^* \left(\left(\frac{d\log(f_t)}{dt} \circ \varphi_*^{-1} \right) \alpha_t \right). \end{aligned}$$

Combining the two identities we get

$$\frac{d\alpha_t}{dt} + i_{X_t} d\alpha_t + d(i_{X_t}\alpha_t) = \underbrace{\left(\frac{d\log(f_t)}{dt} \circ \varphi_*^{-1} \right)}_{=:g_t} \alpha_t.$$

We try to solve this equality with $X_t \in \xi_t = \ker(\alpha_t)$ and a function g_t. If the one forms are evaluated on R_t, the Reeb vector field of α_t, then one gets

$$g_t = \frac{d\alpha_t}{dt} + d(i_{X_t}\alpha_t).$$

This means that if we find a vector field X_t so that $\frac{d\alpha_t}{dt} + i_{X_t} d\alpha_t + d(i_{X_t}\alpha_t)$ is a multiple of α_t, then the factor g_t can be determined from X_t and α_t.

By the contact condition ($d\alpha_t$ is non-degenerate on ξ_t) there is a unique vector field X_t tangent to ξ_t so that

$$\frac{d\alpha_t}{dt} + i_{X_t} d\alpha_t = 0$$

when evaluated on tangent vectors to ξ_t. But this means that $\frac{d\alpha_t}{dt} + i_{X_t} d\alpha_t + d(i_{X_t}\alpha_t)$ is a multiple of α_t. The flow of X_t is a vector field whose flow has the desired properties. (The flow exists since X_t, just like $\dot{\alpha}_t$, has compact support.)

- **Consequence:** Contact structures on closed manifolds are therefore stable in the sense that C^1-small perturbations of contact structures yield *isomorphic* contact structures. Actually, in dimension 3 the following, stronger statement holds [3]: Every contact structure on a closed 3-manifold has a C^0-neighborhood in the space of hyperplane fields such that all contact structures in that neighborhood are isomorphic to the original one.

2. Lecture on October 23 – Consequences of Gray's theorem, Lutz twist

- **Theorem (Darboux):** Let (M, ξ) be a contact manifold of dimension $2n+1$. Then every point has a neighborhood with coordinates $(z, x_1, y_1, \ldots, x_n, y_n)$ so that $\xi = \ker\left(dz - \sum_i y_i dx_i\right)$.
- **Remark:** This means that locally, in the neighborhood of a point, all contact structures are equivalent to one fixed model (depending only on the dimension).
- **Proof:** Let $p \in M$. We choose a contact form α' and a coordinate system (z', x_i', y_i') so that $\xi(p) = \ker(dz')$. Multiplying z' with a suitable constant and adding a suitable vector from $\xi(p)$ we may assume that $dz' = \alpha'$ at p and the Reeb vector field of α at p is $\partial_{z'}$. Moreover, we can choose a symplectic basis X_i', Y_i' of $\xi(p)$ with respect to $d\alpha'(p)$. Then, after changing the coordinate system, we may assume that the constant vector fields X_i', Y_i' (in terms of the old coordinate system) correspond to coordinate vector fields $\partial_{x_i'}, \partial_{y_i'}$. This implies $\alpha'(p) = \alpha(p)$ and $d\alpha(p) = d\alpha'(p)$ where $\alpha = dz' - \sum_i y_i' dx_i'$.

  Applying the procedure from the previous proof to $\alpha_t = t\alpha + (1-t)\alpha'$ yields a family of vector fields X_t which vanish on at p so that $\varphi_t^*\alpha_t$ is a multiple of α. Here φ_t is the local flow of X_t at p and this is defined since $X_t(p) \equiv 0$. Thus φ_1 defines a contact transformation from $(U, \ker(\alpha_0))$ to $(U, \ker(\alpha_1))$.
- Let (M, ξ) be a contact manifold of dimension $2n+1$ with coorientable contact structure and assume that $L \subset M$ is a submanifold with $TL \subset \xi$. Then $\alpha, d\alpha$ both have to vanish when restricted to L. As $d\alpha$ is a non-degenerate two form on $\xi(p)$ for all p, linear algebra implies that $\dim(L) \leq \dim_{\mathbb{R}}(\xi(p))/2$. If $\dim(L) = n$, then L is called *Legendrian.* In this case, one can show that there is a neighborhood of L is diffeomorphic to the contact structure on the space of 1-jets of real functions on L, i.e.

$$\begin{aligned} J^1(L, \mathbb{R}) = \{f_p : \mathrm{Op}(p) \longrightarrow \mathbb{R} \,|\, p \in M, f_p \text{ smooth}\}/\sim \longrightarrow \mathbb{R} \times L \\ [f_p] \longmapsto (f(p), p). \end{aligned}$$

  with $f_p \sim g_q$ if and only if $p = q$, $f_p(p) = g_q(q)$ and $df_p(p) = dq_p(p)$. This space has a canonical contact form

$$\alpha(X) = df_p(\mathrm{pr}_2(X)) - dz(\mathrm{pr}_{1*}(X))$$

for $X \in T_{[f_p]}J^1(L,\mathbb{R})$. This might be reminiscent from symplectic geometry (Weinsteins theorem on tubular neighborhoods of Lagrangian submanifolds asserts that a neighborhood is symplectomorphic to a neighborhood of the zero section in T^*L with its tautological symplectic structure.)

- Another type of submanifold which is useful in dimension 3 are curves transverse to given contact structure ξ. It is relatively easy to show that every simple closed loop K which is transverse to a contact structure has a tubular neighborhood U so that (U, K, ξ) is contactomorphic to $(S^1\times, D^2, S^1 \times \{0\}, \ker(\alpha_0 = dz + r^2 d\vartheta))$.
- The previous normal form allows to perform the following operation on a contact structure, it is called a 2π-Lutz twist. For $\varepsilon > 0$ let $f, g : [0, \varepsilon] \longrightarrow \mathbb{R}$ be smooth functions so that
  1. $f(t) = 1$ and $g(t) = t^2$ near the boundary of the interval
  2. $fg' - gf' > 0$ (for example, f, g have no common zero, so α_1 defines a plane field)
  3. g has exactly one zero in the interior of the interval

  These conditions do the following
  1. ensures that $\alpha_1 = f(r)dz + g(r)d\vartheta$ is smooth and coincides with α_0 near the boundary of the solid torus $\{r \leq \varepsilon\}$,
  2. α_1 is a contact form,
  3. fixes the path (f, g) in the (punctured) plane up to homotopy relative to the boundary.

  Thus one can replace the contact structure ξ with a contact structure ξ' which coincides with $\ker(\alpha')$ inside of $\{r \leq \varepsilon\}$ and with ξ outside of this domain.

  One important property of this operation is that it is well defined, i.e. up to isotopy the resulting contact structure depends only on the transverse isotopy type of K. Clearly, Gray's theorem is useful for this. Also, it is important to note that ξ' is homotopic to ξ as a plane field.

  It is relatively easy to obtain a homotopy α_t of nowhere vanishing 1-forms α_t interpolating between α_0 and α_1. For this add a small 1-form $h(r)dr$ so that h vanishes close to the boundary of the interval where f, g are fixed, and $h > 0$ where α and α' do not coincide. Now one consider $t\alpha_1 + (1+t)\alpha_0 + h(r)dr$ with $t \in [0, 1]$. This is not a family of contact forms.
- **Fundamental question:** Is ξ' a new contact structure (i.e. not isotopic to ξ), or not.
- **Definition:** Let ξ be a contact structure on a connected 3-manifold. Then ξ is *overtwisted*, if there is an embedded disc D^2 so that ξ is tangent to D^2 along the boundary, D is an *overtwisted* disc. If ξ is not overtwisted, then it is *tight*.
- **Answer to the above question:** $\xi' \simeq \xi$ if and only if ξ is overtwisted.
- Proving this is a bit too much, it is difficult already interesting to prove that some contact structure is tight.
- Clearly, after Lutz twist is applied to a contact structure the result is overtwisted.
- Notice that for $\xi = \ker(dz - ydx)$

$$\begin{aligned} \psi : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (tx, ty, t^2 z) \end{aligned}$$

is a contact transformation for all $t > 0$. Thus if $\mathbb{R}^3$ is overtwisted, then every neighborhood of the origin contains an overtwisted disc. Therefore, if $(\mathbb{R}^3, \ker(dz - ydx))$ is not tight, then no contact structure is.

- **Theorem (Bennequin, 1983** [1]**):** $(\mathbb{R}^3, \ker(dz - ydx))$ is tight.
- The original proof of this theorem is rather involved.

Lecture on October 30 – Tightness criterion, Isotopy extension, transverse knots and braids Bennequin theorem preliminaries

- The following theorem is the most frequently used criterion to ensure that a given contact structure is tight.
- **Definition:** Let (M, ξ) be a closed 3-manifold with a contact structure ξ. A compact symplectic 4-manifold (X, ω) is a *weak symplectic filling* if
  1. (M, ξ) with the contact orientation is one component of ∂X with its orientation as boundary (outward normal first convention),
  2. $\omega|_\xi$ is non-degenerate.
- **Theorem (Eliashberg):** If (M, ξ) is symplectically fillable, then it is tight.
- Notice that (B^4, ω_{st}) is a symplectic filling of $(S^3 \subset \mathbb{C}^2, \xi = TS^3 \cap iTS^3)$. There for this contact structure is tight and so is the standard contact structure on $\mathbb{R}^3$.
- Let (M, ξ) be a contact manifold, α a contact form, and consider vector field X with compact support whose flow preserves ξ (i.e. $L_X\alpha$ is a multiple of α) Then
$$\begin{aligned}\{\text{contact vector fields on } M\} &\longrightarrow C^\infty_{cpt}(M)\\ X &\longmapsto \alpha(X)\end{aligned}$$
is a linear map which has in inverse: Let R_α be the Reeb vector field of α. Again, this is a consequence of the fact that $d\alpha$ is non-degenerate on ξ. For $f \in C^\infty_{cpt}(M)$ we seek a vector field X_f tangent to ξ so that $fR_\alpha + X$ is a contact vector field, i.e. we seek X Legendrian so that
$$L_{fR_\alpha + X}\alpha = df + i_X d\alpha$$
vanishes on ξ. Thus, we want to solve $i_X d\alpha|_\xi = -df|_\xi$, but this has a unique solution.
- A consequence of this is the isotopy extension property: Let $\varphi_t : \mathrm{Op}(N) \longrightarrow M$ be a contact isotopy defined on the neighborhood of a submanifold N. Then there is a contact isotopy $\psi_t : M \longrightarrow M$ of M which coincides with φ_t on a smaller neighborhood of N. To see this, not that φ_t gives rise to a time dependent velocity vector field near N which is a contact vector field. Using the correspondence between smooth, compactly supported functions and contact vector fields we obtain a family f_t of smooth functions near $\varphi_t(N)$. Use cut-off functions to obtain a family of globally defined, smooth functions (with compact support) g_t so that $g_t \equiv f_t$ near $\varphi_t(N)$. The flow of the resulting family of global contact vector fields is the desired isotopy ψ_t.
- **Corollary:** Let $n > 0$ and (M, ξ) a path connected contact manifold of dimension $2n+1$. For each pair of k-tuples $(p_1, \ldots, p_k), (p'_1, \ldots, p'_k)$ there is a contact transformation φ of M so that $\varphi(p_j) = p'_j$.
- **Proof:** One chooses k pairwise disjoint paths in M connecting p_i to p'_i. These paths can be covered by Darboux charts. In this charts one can push points around and extend the isotopy using the isotopy extension property.

- **Corollary:** Let K, K' be two closed curves in $\mathbb{R}^3$ transverse to a contact structure ξ which are isotopic through transverse knots. Then K, K' are contact isotopic.
- **Definition:** Let ξ be a cooriented contact structure. Then a link K is positively transverse of $\alpha|_K$ represents the orientation of K for a defining form of a contact structure ξ representing the coorientation of ξ.
- The proof of Bennequin's theorem uses various notions like knots which are adapted to the plane field and other structures. We first review these notions
- **Reminder:** A *knot* is an embedding of an oriented circle in $\mathbb{R}^3$. We usually consider oriented knots. Knots are often considered equivalent is they are isotopic. An (oriented) *link* is a disjoint union of (oriented), two links are equivalent if they are isotopic. For every oriented knot/link there is a compact oriented surface with boundary whose oriented boundary is the link/knot. Such a surface is called *Seifert surface* (and closed components are usually discarded.)

  The genus of a knot/link is the minimal genus of a Seifert surface. In the case of a link, it is easier to consider the Euler characteristic of a surface rather than the genus.
- **Reminder:** Let Σ be a surface. The *genus* g of Σ is the maximal number of pairwise disjoint simple closed curves such that removing these loops does not increase the number of connected components. If Σ is compact with boundary, l is the number of boundary components and c is the number of connected components, then the Euler characteristic $\chi(\Sigma)$ of Σ is defined and satisfies

$$\chi(\Sigma) = 2c - 2g - l.$$

- **Reminder:** A knot/link is a *closed braid* if it is disjoint from the z-axis and positively transverse to the half-planes $\{\vartheta = \vartheta_0\} \subset \mathbb{R}^3 \setminus \{(0,0,z) \,|\, z \in \mathbb{R}\}$ for $\vartheta_0 \in [0, 2\pi]$ (cooriented by ∂_ϑ). The equivalence relation for closed braids is isotopy through closed braids. According to a theorem of Alexander, every link is isotopic to a closed braid. Moreover, according to a theorem of Markov, if two braids are isotopic as links, then the two braids are isotopic as braids after stabilization.
- **Theorem:** Let $K \subset \mathbb{R}^3$ be a link positively transverse to $\xi = \ker(dz + \rho^2 d\vartheta)$ Then K is isotopic as a transverse knot to a closed braid.
- **Proof:** After small isotopy, we may assume that K is disjoint from the z-axis. Let $\dot{z}, \dot{\rho}, \dot{\vartheta}$ denote the components of the positive unit tangent vector to K. Now K can be decomposed into the good part, where $\dot{\vartheta} > 0$, and the complement (which is bad).

  Note that $\dot{z} + \rho^2\dot{\vartheta} > 0$ along K since K is positively transverse. If $\gamma \subset K$ is a bad piece, then $dz > 0$ along γ, i.e. the z-coordinate increases along γ. In particular, K cannot be entirely bad because it is closed.

  The following standard perturbation of a bad piece will be used frequently: On an interval in a bad segment keep the z- and ρ-components fixed but replace the ϑ component (which is monotonically decreasing originally) by a function $\widehat{\vartheta}$ which coincides with ϑ and is increasing near a specified point of the segment. If $\widehat{\vartheta}$ is chosen carefully enough (with a lower bound on $\dot{\widehat{\vartheta}} - \dot{\vartheta}$), then the resulting curve is still transverse and isotopic through transverse curves.

  After applying the standard perturbation sufficiently often, one may assume that the variation of ϑ along a bad piece is smaller than 2π. If there is now other piece of γ lying between the z-axis and a bad piece along a radial line, then we hope to isotope the bad piece along radial lines through transverse curves so that the result contains fewer bad arcs.

We want to arrange that no segment of K lies between a bad arcs and the z-axis along a radial line. By transversality, we may assume that there are only finitely many points of K lying between a bad segment and the z-axis. Moreover, we require $\dot{\vartheta} \neq 0$ at those points. This leads to four possible configurations (a crossing of oriented arcs in the ϑ, z-plane). One of these configuration is excluded for transverse knots. The other three can be modified be standard modifications along a bad arc so that for the resulting curve no radial segment between a point on a bad arc and the z-axis meets the transverse link in its interior.

References

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