COMBINATORIAL TORSIONS IN TOPOLOGY

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BLOCK SEMINAR, MARCH 27 – MARCH 31 2023

1. CONTENT

Torsions were introduced to algebraic/geometric topology by Reidemeister, who used them to distinguish lens spaces up to homeomorphism and homotopy equivalence. An important consequence of the properties of these invariants used is that homotopy equivalence can be strengthened to simple homotopy equivalence, which is better suited for a combinatorial approach to homotopy theory which was envisioned by Whitehead.

In this seminar, we begin with algebraic constructions of torsions and the Whitehead group. This will be applied to finite CW-complexes. Using these results we will obtain a classification of lens spaces up to homotopy, duality theorems and results about the torsion of 3-manifolds

Target Audience: Students of Mathematics, TMP

Prerequisites: Some exposure to algebraic topology, modules over rings.

REFERENCES

[Co] M. Cohen, *A course in simple homotopy theory*, Springer GTM 10

[Mi] J. Milnor, *Whitehead torsion*, Bull. AMS 72 (1966), 358-426.

[Tu] V. Turaev, *Introduction to combinatorial torsion*, Birkäuser.

[Mn] P. Mnev, Lecture notes on torsions, https://arxiv.org/pdf/1406.3705.pdf.

[Ni] L. Nicolaescu, Notes on Reidemeister torsion

[SZ] R. Stöcker, H. Zieschang, *Algebraische Topologie*, Teubner 1994.

2. TALKS

Many things are missing: Application to the Hauptvermutung, analytic torsions, Chapman's theorem in topological invariance...

(1) **Homotopy and related concepts, Whiteheads combinatorial approach**
- Date:
- Speaker:
- Literature: [Co], p.2–4, [SZ]
- Discuss homotopy, retractions, homotopy equivalences, mapping cones etc., and introduce the notion of *simple homotopy equivalence*

(2) **CW-complexes, cellular homology**
- Date:
- Speaker:
- Literature: [SZ]
- Discuss definitions CW-complexes and their topological properties. Mention cellular approximation and define the cellular chain complex. This will be out main example of a chain complex with distinguished generating set.

(3) **Torsion of chain complexes**
- Date/Time:
- Speaker:
- Literature: [Tu], p.1–10
- Define the torsion of an acyclic chain complex of vector spaces of finite dimension and preferred basis (c.f. [Co] p.54–57) Discuss multiplicativity and duality. Describe the computation of torsion using matrix τ-chains.)

(4) **Methods for computation**
- Date:
- Speaker:
- Literature: [Tu], p.10–16.
- Explain how torsion can be computed from a chain contraction. (Section 2.2 in [Tu], for the proof see [Co], p.52–54.) Generalise the definition of torsion to finitely generated modules over commutative rings with unit.

(5) **Whitehead groups**
- Date:
- Speaker:
- Literature: [Co], p.36–44.
- Discuss the Whitehead group $K_G(R)$ in more detail for some rings R. For example $K_1(\mathbb{Z}) = \{0\}$, $K_1(\mathbb{Z}_5) \neq 0$ (this is on p.33 of [Co]).

(6) **Characterisation of torsion of a chain complex, change of rings**
- Date:
- Speaker:
- Literature: [Co], p.56–61.
- Prove that the torsion is characterized by few properties and effects of a change of rings. Recall the universal coefficient theorem for homology (which allows to compute the homology of $C \otimes R'$ from the homology of $H_*(C, R)$ and $\mathrm{Tor}(R, R')$.)

(7) **Covering spaces**
- Date:
- Speaker:

- Literature: [Co], p.9–13.
- Survey the classification of coverings for sufficiently connected spaces. Introduce the universal covering and outline the CW-structure on the universal cover of a CW-complex X. Explain that the cellular chain complex is a $\mathbb{Z}[\pi(X)]$-chain complex.

(8) **Reidemeister-Franz torsion, Whitehead torsion**
- Date:
- Speaker:
- Literature: [Tu], p.30–38.
- Define the Reidemeister-Franz torsion of a CW-complex, show that it is well defined under cellular subdivision, compute the torsion of S^1. Define the Whitehead torsion of deformation retracts and prove the realization property.

(9) **Whitehead torsion, simple homotopy equivalences**
- Date:
- Speaker:
- Literature: [Tu], p.38–43.
- Continue the previous talk: Discuss the Whitehead torsion of a homotopy equivalence, simple homotopy equivalences, and how homotopy equivalences interact with Reidemeister-Franz torsion.

(10) **Lens spaces**
- Date:
- Speaker:
- Literature: [Tu], p.44–51.
- Compute the torsion of a lens space and discuss the classification of lens spaces.

(11) **Milnor torsion**
- Date:
- Speaker:
- Literature: [Tu], p.51–57.
- Define the Milnor torsion and compute it for 3-manifolds

(12) **Torsions of manifolds – one or two talks**
- Date: one or two talks
- Speaker:
- Literature: [Tu], p.69 –81.
- Briefly introduce triangulations of manifolds and prove the duality theorem for torsions. It might be useful to discuss the proof of Poincaré duality based on triangulations before in a separate talk.

(13) **Application to knots – one or two talks**
- Date:

- Speaker:
- Literature: [Tu], p.81–95.
- Define the Alexander polynomial and its relations ship to Milnor torsion. Compute the Alexander polynomial using the Fox differential calculus. Outline how the torsion of a 3-manifold obtained by surgery on a link can be computed from the Alexander polynomial.