Ludwig-Maximilians-Universität München
Prof. Dr. Thomas Vogel

<center>

Seminar
Summer term 2019

Atiyah-Singer index theorem

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In this seminar we will discuss a proof of a version of the Atiyah-Singer index theorem concerning Dirac operators. In order to define all objects we will discuss basics from the representation theory of Clifford algebras, analytic properties of the Dirac operator, and the Chern-Weil description of differential forms representing characteristic classes of vector bundles.

The proof is based on asymptotic properties of fundamental solutions of the heat equation, and we will use [R] almost exclusively. Other proofs used cobordism theory ([Pa]) or K-theory (cf. [AS, Sh]).

In order to illustrate the methods of the proof we will obtain a variety of other important results (Hodge decomposition, results on the spectrum of the Laplace operator, and the Lefschetz formula computing the algebraic number of fixed points of a smooth map of a closed manifold to itself).

Applications of the theorem include obstructions for the existence of positive scalar curvature metrics.

Audience: Master in Math., TMP

Prerequisites: Necessary: Manifolds, Riemannian geometry. Helpful: Vector bundles, analysis on manifolds

Time & place: Wednesday, 12-14am, Room B045

Organizational meeting: Wednesday, April 24, 8:30–10am, Room B252

<center>REFERENCES</center>

[AS]   M. Atiyah, I. Singer, *The index of elliptic operators I,*
[BB]   B. Boos, D. Bleecker, *Topology and analysis*, Springer Universitext 1985.
[Hu]   D. Huybrechts, *Complex geometry*, Springer Universitext 2004.
[LM]   H. B. Lawson, M. L. Michelsohn, *Spin geometry*, Princeton Univ. Press 1989.
[Mi]   J. Milnor, *Characteristic classes*, Annals of Mathematics Studies 76, Princeton Univ. Press. 1974.
[Pa]   R. Palais, *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, No. 57, Princeton Univ. Press 1965.
[R]   J. Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Research notes in Math. 179, Longman Scientific & Technical 1993.
[Sh]   P. Shanahan, *The Atiyah-Singer index theorem*, Springer, Lecture notes in Mathematics 638, Springer 1978.

Atiyah-Singer index theorem
Programm

May 8   Clifford algebras and Dirac operators
Literature: [R] Chapter 2, p. 25–36.
Speaker: J.V.
            Clifford algebra, Clifford bundles, Dirac operator, Weitzenboeck formula,
            Examples of Clifford bundles (Exterior bundle as Clifford bundle with $d + d^*$
            as Dirac operator, the Kaehler analogue with $(0, q)$-forms and $\overline{\partial} + \overline{\partial}^*$ as
            Dirac operator), Spin^c-structure.
            For the last example, some formalism related to complex structures is used.
            A reference are Sections 1.2–1.3 of [Hu].

May 15   Analytic properties of the Dirac operator 1
Literature: [R] Chapter 3, p. 37 – 45.
Speaker: T.V.
            Sobolov spaces, Garding inequality, ellipticity of D, elliptic estimate

May 22   Analytic properties of the Dirac operator 2, Hodge theory
Literature: [R], Chapter 3 & 4, p. 46–57 (starting from Definition 3.19)
Speaker: L.G.
            Smoothing properties, spectral theory for the Dirac operator,
            Hodge: harmonic representatives of de Rham cohomology classes,
            Poincaré-duality
            This is one of the first applications of analysis to algebraic topology.

May 29   Heat equation, wave equation 1
Literature: [R] Chapter 5, p. 58–67 until before Definition 5.15
Speaker: V.N.
Vincent Neeb   heat equation, fundamental solution

June 5   Heat equation, wave equation 2
Literature: [R] Chapter 5, p. 67 –79 starting with Definition 5.15
Speaker: M.P.
            asymptotic expansion of the heat kernel, zeroth and first coefficient
            of the expansion
            Theorem 5.16 about the shape of the asymptotic expansion for the heat
            kernel is fundamental.

June 12   Trace class operators, spectral geometry
Literature: [R] Chapter 6,7, p. 80–94 (as far as one can get)
Speaker: M.T.
            Chapter 6 is a summary of standard material on Hilbert-Schmidt operators
            and trace class operators. Chapter 7 is an application of the asymptotic
            expansion of the heat kernel to the study of the spectrum of the Laplace
            operator: The Weyl asymptotic for the n-th eigenvalue of a closed
            Riemannian manifold. (This is a luxury.)

June 19   Lefschetz formula
Literature: [R] Chapter 8, p. 95–103
Speaker: L.N.
            analytic proof of the Lefschetz fixed point formula
            illustrates the method used later to prove Atiyah-Singer index thm.:
            It is based on a comparison of heat kernel for different time
            parameters ($t \to \infty$ and $t \to 0$)

June 26	Fredholm Index
Literature:	[R] Chapter 9 p. 104–109
Speaker:	L.B.
	Index of a graded Dirac operator, its connection to long time behavior
	of the heat kernel, and as integral of the $n/2$-coeff. of the asymptotic expansion
July 3	Chern-Weyl approach to characteristic classes, spin structures
Literature:	[R] Chapter 10A-10B, p. 110–123, cf. [Mi] Appendix C.
Speaker:	Z.G.
	Characteristic classes as integrals of polynomials in curvature $\mathcal{L}$-genus,
	Chern classes/character , $\widehat{A}$-genus,
July 10	Spin structures, Harmonic oscillator
Literature:	[R] Chapter 10B-10C, p. 124–136
Speaker:	
	Spin structures induce particular Clifford bundles which
	have an additional grading
July 17	Index theorem
Literature:	[R] Chapter 11 p.136–147
Speaker:	L.B.
	Computation of the relevant coefficient of the heat kernel expansion in terms of
	characteristic classes
July 24	positive scalar curvature, Hirzebruch signature theorem, Γ-index theorem
Literature:	[R] Chapter 11 and 13, p. 147–152, p. 168–176
Speaker:	A.S.
	non-vanishing $\widehat{A}$-genus obstructs positive scalar curvature metrics on
	Spin manifolds, computation of the signature in terms of the $\mathcal{L}$-genus,...
	as time permits