Mathematisches Institut LMU https://www.math.lmu.de/~nam/Numerics24.php
Phan Thành Nam, Giao Ky Duong SoSe 2024

Numerics II

Homework Sheet 3

(Released 2.5.2024 – Discussed 7.5.2024)

E3.1 Show that $C^1(0,1)$ is not dense in L^∞, with L^∞ norm.

E3.2 Let $f \in W^{1,1}(0,1) \cap C([0,1])$. Assume that we can find $\{f_n\} \subset C_c^\infty(0,1)$ such that $f_n \to f$ in $W^{1,1}(0,1)$. Prove that $f(0) = f(1) = 0$. Deduce that $C_c^\infty(0,1)$ is not dense in $W^{1,1}(0,1)$.

E3.3 Construct a function in $W^{1,2}(0,1)$, but not in $W^{2,2}(0,1)$ and explain the answer.

E3.4 Let $\Omega = \{(x,y) \in \mathbb{R}^2, 0 < x < 1, |y| < x^{10}\}$ and define

$$u(x,y) = x^{-\frac{1}{2}}.$$

(i) Prove that $u \in L^2(\Omega)$.

(ii) Prove that the weak derivatives $\partial_x u$, $\partial_y u$ exist and determinate them.

(iii) Prove that $u \in W^{1,2}(\Omega)$ and $u \notin L^\infty(\Omega)$.

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Homework Sheet 2

(Released 24.4.2024 – Discussed 30.4.2024)

E2.1 In the lecture, we proved that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} f\varphi = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ then $f = 0$ a.e. Prove the same result with $\mathbb{R}^d$ replaced by a general open subset $\Omega \subset \mathbb{R}^d$.

E2.2 In this exercise we consider functions mapping from $\Omega \to \mathbb{R}$ with $\Omega = (-1, 1)$.

(i) Prove that $f(x) = |x|^{\frac{3}{2}} - 1$ has a weak derivative in $L^1_{\text{loc}}(\Omega)$.

(ii) Is there a weak derivative in $L^1_{\text{loc}}(\Omega)$ of the following function?

$$g(x) = \begin{cases} x & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

E2.3 Consider the Cantor set $C_\infty = \bigcap_{k=1}^\infty C_k$, where C_k satisfy the following recurrence

$$C_0 = [0, 1], \quad C_{k+1} = \frac{1}{3}C_k \cup \left\{ \frac{2}{3} + \frac{1}{3}C_k \right\}, k \geq 1.$$

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0, 1] \backslash C_\infty, \\ 2x + 1 & \text{if} \quad x \in C_\infty. \end{cases}$$

(i) Find $\sup f, \inf f$, and $\|f\|_{L^\infty}$.

(ii) Find its weak derivative f' and compute

$$f(x) - \int_0^x f'(s) ds.$$

E2.4 Let $u \in L^1_{\text{loc}}(\mathbb{R})$ which has a second order weak derivative $f'' \in L^1_{\text{loc}}(\mathbb{R})$. Show that it has also the first order weak derivative $f' \in L^1_{\text{loc}}(\mathbb{R})$.

E2.5 Let $f \in W^{1,p}(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi(x) = 1$ if $|x| \leq 1$. Denote $f_n(x) = \chi(nx) f(x)$. Prove that

$$f_n \to f \text{ in } W^{1,p}(\mathbb{R}^d) \text{ as } n \to +\infty.$$

Recall that $\|f\|^p_{W^{1,p}} = \|f\|^p_{L^p} + \sum_{|\alpha|=1} \|D^\alpha f\|^p_{L^p}$.

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Numerics II

Homework Sheet 1

(Released 18.4.2024 – Discussed 23.4.2024)

E1.1 Define

$$V = \{u \in L^2(0,1) \,|\, a(u,u) < \infty \text{ and } u(0) = 0\}, \quad a(u,v) = \int_0^1 u'(x)v'(x)\mathrm{d}x.$$

(i) Prove the Cauchy-Schwarz inequality

$$|a(u,v)| \leq \sqrt{a(u,u)}\sqrt{a(v,v)}, \quad \forall u,v \in V.$$

When does the equality hold?

(ii) Define $\|u\|_E = \sqrt{a(u,u)}$. Verify that $\|\cdot\|_E$ is a norm. If we don't have the condition $u(0) = 0$, is $\|\cdot\|_E$ a norm?

(iii) Prove the "Parallelogram law"

$$\|u\|_E + \|v\|_E = \frac{1}{2}\left[\|u+v\|_E^2 + \|u-v\|_E^2\right] \quad \forall u,v \in V.$$

E1.2 Based on the lecture, formulate the weak formulation of the following problem

$$\begin{cases} -u'' + u &= f, \\ u(0) = u(1) &= 0, \end{cases}$$

where $f \in L^2[0,1]$.

E1.3 Let V and $a(u,v)$ be given in E1.1. Prove that for $u \in V \cap C^1[0,1]$ we have

(i) $\|u\|^2_{L^2(0,1)} \leq \frac{1}{2}\|u'\|^2_{L^2(0,1)}$

(ii) $\|u\|^2_{L^2(0,1)} \leq \frac{1}{\sqrt{8}}\|u'\|^2_{L^2(0,1)}$ if we assume further $u(1) = 0$

(iii) $\|u\|^2_{L^2(0,1)} \leq \frac{1}{6}\|u'\|^2_{L^2(0,1)}$ if we assume further $\int_0^1 u = 0$

(iv) $\max_{x\in[0,1]} |u(x)|^2 \leq 2u^2(1) + 2\|u'\|^2_{L^2}$

(v) $\max_{x\in[0,1]} |u(x)|^2 \leq 2(\|u\|^2_{L^2} + \|u'\|^2_{L^2})$