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The Unitary Group in Its Strong Topology

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Abstract

The goal of this paper is to confirm that the unitary group $\mathrm{U}(\mathcal{H})$ on an infinite dimensional complex Hilbert space $\mathcal{H}$ is a topological group in its strong topology, and to emphasize the importance of this property for applications in topology. In addition, it is shown that $\mathrm{U}(\mathcal{H})$ in its strong topology is metrizable and contractible if $\mathcal{H}$ is separable. As an application Hilbert bundles are classified by homotopy.

Keywords

Unitary Operator, Strong Operator Topology, Topological Group, Infinite Dimensional Lie Group, Contractibility, Hilbert Bundle, Classifying Space

1. Introduction

The unitary group $\mathrm{U}(\mathcal{H})$ plays an essential role in many areas of mathematics and physics, e.g. in representation theory, number theory, topology and in quantum mechanics. In some of the corresponding research articles complicated proofs and constructions have been introduced in order to circumvent the assumed fact that the unitary group is not a topological group when equipped with the strong topology (see Remark 1 below for details). However, in Proposition 1 it is proven that $\mathrm{U}(\mathcal{H})$ is indeed a topological group with respect to the strong topology. Moreover, in this paper it is shown that the compact open topology and the strong topology agree on $\mathrm{U}(\mathcal{H})$, and that this topology is metrizable and contractible if $\mathcal{H}$ is separable (and infinite dimensional). To demonstrate the relevance of these topological considerations it is shown that these results lead to a straightforward classifications of Hilbert bundles. Furthermore, the possibility of finding a Lie structure on $\mathrm{U}(\mathcal{H})$ with respect to the strong topology is discussed in a new line the following header.

2. The Unitary Group as a Topological Group

It is easy to show and well-known that the unitary group $\mathrm{U}(\mathcal{H})$—the group of all unitary operators $\mathcal{H} \to \mathcal{H}$ on a complex Hilbert space $\mathcal{H}$—is a topological group with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. However, for many purposes in mathematics the norm topology is too strong. For example, for a compact topological group G with Haar measure μ the left regular representation on $\mathcal{H} = L_2(G,\mu)$

$$L : G \to \mathrm{U}(\mathcal{H}), g \mapsto L_g : \mathcal{H} \to \mathcal{H}, L_g f(x) = f(g^{-1}x) \tag{1}$$

is continuous for the strong topology on $\mathrm{U}(\mathcal{H})$, but L is not continuous when $\mathrm{U}(\mathcal{H})$ is equipped with the norm topology, except for finite G. This fact makes the norm topology on $\mathrm{U}(\mathcal{H})$ useless in representation theory and its applications as well as in many areas of physics or topology. The continuity property which is mostly used in case of a topological space W and a general Hilbert space $\mathcal{H}$ and which seems to be more natural is the continuity of a left action of W on $\mathcal{H}$

$$\Phi : W \times \mathcal{H} \to \mathcal{H}, \tag{2}$$

in particular, in case of a left action of a topological group G on $\mathcal{H}$: Note that the above left regular representation is continuous as a map: $L : G \times \mathcal{H} \to \mathcal{H}$.

Whenever Φ is a unitary action (*i.e.* $\hat{\Phi}(w) : f \mapsto \Phi(w,f)$ is a unitary operator $\hat{\Phi}(w) \in \mathrm{U}(\mathcal{H})$ for all $w \in W$) the continuity of Φ is equivalent to the continuity of the induced map

$$\hat{\Phi} : W \to \mathrm{U}(\mathcal{H}) \tag{3}$$

with respect to the strong topology on $\mathrm{U}(\mathcal{H})$. In fact, if the action Φ is continuous then $\hat{\Phi}$ is strongly continuous by definition of the strong topology. The converse holds since $\mathrm{U}(\mathcal{H})$ is a uniformly bounded set of operators. The corresponding statement for the general linear group $\mathrm{GL}(\mathcal{H})$ of bounded invertible operators holds for the compact open topology on $\mathrm{GL}(\mathcal{H})$ instead of the strong topology. On $\mathrm{U}(\mathcal{H})$ the two topologies coincide, see Proposition 2 below.

We come back to the continuity of unitary actions in a broader context at the end of this paper where we elucidate the significance of the fact that $\mathrm{U}(\mathcal{H})$ is a topological group for the classification of Hilbert bundles over paracompact spaces X.

Proposition 1: $\mathrm{U}(\mathcal{H})$ *is a topological group with respect to the strong topology.*

Proof. Indeed, the composition $(S,T) \mapsto ST$ is continuous: Given $(S_0,T_0) \in \mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})$ let $\mathcal{V}$ be a neighbourhood of S_0T_0 of the form $\mathcal{V} = \left\{R \in \mathrm{U}(\mathcal{H}) : \left\|(R - S_0T_0)f\right\| < \varepsilon\right\}$ where $f \in \mathcal{H}$ and $\varepsilon > 0$. Now,
$\mathcal{W} := \left\{(S,T) \in \mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H}) : \left\|(S - S_0)T_0 f\right\| < \frac{1}{2}\varepsilon, \left\|(T - T_0)f\right\| < \frac{1}{2}\varepsilon\right\}$
is a neighbourhood of $(S_0,T_0) \in \mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})$ and for $(S,T) \in \mathcal{W}$ we have:

$$\left\|(ST - S_0T_0)f\right\| \le \left\|S(T - T_0)f\right\| + \left\|(ST_0 - S_0T_0)f\right\| \tag{4}$$

$$\le \left\|(T - T_0)f\right\| + \left\|(S - S_0)T_0 f\right\| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \tag{5}$$

i.e. $\{ST:(S,T)\in\mathcal{W}\}\subset\mathcal{V}$. To show that $T\mapsto T^{-1}$ is continuous in T_0 let $\mathcal{V}=\{S\in \mathrm{U}(\mathcal{H}):\|(S-T_0^{-1})f\|<\varepsilon\}$ a typical neighbourhood of T_0^{-1} in $\mathrm{U}(\mathcal{H})$. For $g:=T_0^{-1}f$ let $T\in \mathrm{U}(\mathcal{H})$ satisfy $\|(T-T_0)g\|<\varepsilon$. Then

$$\|(T^{-1}-T_0^{-1})f\|=\|T^{-1}T_0g-g\|=\|T_0g-Tg\|<\varepsilon, \tag{6}$$

i.e. $\{T^{-1}:\|(T-T_0)g\|<\varepsilon\}\subset\mathcal{V}$. ■

Remark 1. This result with its simple proof is only worthwhile to publish because in the literature at several places the contrary is stated and because therefore some extra but superfluous efforts have been made. For example, Simms [1] explicitly states that the unitary group is not a topological group in its strong topology and that therefore the proof of Bargmann's theorem [2] has to be rather involved. But also recently in the paper of Atiyah and Segal [3] some proofs and considerations are overly complicated because they assume that the unitary group is not a topological group[1]. The assertion of proposition 1 has been mentioned in [4].

The misunderstanding that $\mathrm{U}(\mathcal{H})$ is not a topological group in the strong topology might come from the fact that the composition map

$$\mathrm{B}(\mathcal{H})\times\mathrm{B}(\mathcal{H})\to\mathrm{B}(\mathcal{H}),(S,T)\mapsto ST, \tag{7}$$

is not continuous in the strong topology (where $\mathrm{B}(\mathcal{H})$ denotes the space of bounded linear operators) and consequently $\mathrm{GL}(\mathcal{H})$ is not a topological group with respect to the strong topology (in the infinite dimensional case). But the restriction of the composition to $\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H})$ is continuous since all subsets of $\mathrm{U}(\mathcal{H})$ are uniformly bounded and equicontinuous.

Another assertion in [3] is that the compact open topology on $\mathrm{U}(\mathcal{H})$ is strictly stronger than the strong topology[2] and therefore some efforts are made in [3] to overcome this assumed difficulty. However, again because of the uniform boundedness of the operators in $\mathrm{U}(\mathcal{H})$ one can show:

Proposition 2: *The compact open topology on* $\mathrm{U}(\mathcal{H})$ *coincides with the strong topology.*

Proof. The compact open topology on $\mathrm{B}(\mathcal{H})$ and hence on $\mathrm{U}(\mathcal{H})$ is generated by the seminorms $T\mapsto\|T\|_K:=\sup\{\|Tf\|:f\in K\}$ where $K\subset\mathcal{H}$ is compact. Let $\mathcal{V}=\{T\in\mathrm{U}(\mathcal{H}):\|T-T_0\|_K<\varepsilon\}$ be a typical neighbourhood of $T_0\in\mathrm{U}(\mathcal{H})$ where $K\subset\mathcal{H}$ is compact and $\varepsilon>0$. We have to find a strong neighbourhood $\mathcal{W}$ of T_0 such that $\mathcal{W}\subset\mathcal{V}$. Let $\delta:=\frac{1}{3}\varepsilon$. By compactness of K there is a finite subset $F\subset\mathcal{H}$ such that $K\subset\bigcup\{B(f,\delta):f\in F\}$ where $B(f,r)=\{g\in\mathcal{H}:\|f-g\|<r\}$ is the usual open ball around f of radius r. Now, for $k\in K$ there exist $f\in F$ with $k\in B(f,\delta)$ and $g\in B(0,\delta)$ such that $k=f+g$. We conclude, for $\|T-T_0\|_F<\delta$

$$\|(T-T_0)k\|\le\|(T-T_0)f\|+\|(T-T_0)g\|<\delta+2\delta=\varepsilon. \tag{8}$$

[1]Explicitly stated in Appendix 1 of [3].

[2]In the beginning of Section 2 of [3] and in the Appendix 1.

As a consequence, the strongly open $\mathcal{W} = \{T \in \mathrm{U}(\mathcal{H}) : \|T - T_0\|_F < \delta\}$ is contained in $\mathcal{V}$. ■

Corollary: *The group* $\mathrm{U}(\mathcal{H})$ *with the strong topology acts continuously by conjugation on the Banach space* $\mathcal{K}(\mathcal{H})$ *of compact operators.*

This follows from the corresponding result [3] (Appendix 1, A1.1) for the compact open topology or it can be shown as in the proof of Proposition 1 using equicontinuity.

The proof of proposition 2 essentially shows that on an equicontinuous subset W of $\mathrm{B}(\mathcal{H})$ the strong topology is the same as the compact open topology. Furthermore, both topologies coincide on W with the topology of pointwise convergence on a total subset $D \subset \mathcal{H}$.

In particular, if $\mathcal{H}$ is separable with orthonormal basis $(e_k)_{k\in\mathbb{N}}$, the seminorms $T \mapsto \|Te_k\|$ generate the strong topology. A direct consequence is (in contrast to an assertion in Wikipedia[3] which explicitly presents $\mathrm{U}(\mathcal{H})$ with respect to the strong topology as an example of a non-metrizable space):

Proposition 3: *The strong topology on* $\mathrm{U}(\mathcal{H})$ *is metrizable*[4] *if* $\mathcal{H}$ *is separable.*

The remarkable result of Kuiper [5] that $\mathrm{U}(\mathcal{H})$ is contractible in the norm topology if $\mathcal{H}$ is infinite dimensional and separable is true also with respect to the compact open topology (see e.g. [3]). By proposition 2 we thus have

Corollary: $\mathrm{U}(\mathcal{H})$ *is contractible in the strong topology if* $\mathcal{H}$ *is infinite dimensional and separable.*

Remark 2. The first three results extend to the projective unitary group $\mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/\mathrm{U}(1) \cong \mathrm{U}(\mathbb{P}\mathcal{H})$: This group is again a topological group in the strong topology, the strong topology coincides with the compact open topology and it is metrizable for separable $\mathcal{H}$. Moreover we have the following exact sequence of topological groups

$$1 \to \mathrm{U}(1) \to \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H}) \to 1 \qquad (9)$$

exhibiting $\mathrm{U}(\mathcal{H})$ as a central extension of $\mathrm{PU}(\mathcal{H})$ by $\mathrm{U}(1)$ in the context of topological groups and at the same time as a U(1)-bundle over $\mathrm{PU}(\mathcal{H})$.

Using the homotopy sequence associated to (9), $\mathrm{PU}(\mathcal{H})$ turns out to be simply connected (with respect to the strong topology). And $\mathrm{PU}(\mathcal{H})$ is an Eilenberg-MacLane space $K(\mathbb{Z},2)$. Hence, $\mathrm{PU}(\mathcal{H})$ is not contractible. (Recall that for natural numbers $n \in \mathbb{N}$ an Eilenberg-MacLane space $K(\mathbb{Z},n)$ is a topological space X whose n^{th} homotopy group $\pi_n(X)$ is isomorphic to $\mathbb{Z}$ whereas all other homotopy groups $\pi_k(X)$ are zero.)

The above sequence (9) is not split as an exact sequence of topological groups or as an exact sequence of groups. Moreover, one can show that even a continuous section $\mathrm{PU}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})$ does not exist [4]: Every section is

[3]See revision (07.11.13): https://en.wikipedia.org/w/index.php?title=Metrization_theorem&oldid=580602815

[4]In an earlier version of this note we claimed that $\mathrm{U}(\mathcal{H})$ is complete. We thank D. Buchholz for pointing out that this is false.

neither continuous nor a group homomorphism.

3. Search for a Lie Group Structure

In view of the result of proposition 1 it is natural to ask whether $\mathrm{U}(\mathcal{H})$ has the structure of a Lie group with respect to the strong topology. Let us review what happens in the case of the norm topology:

We know that $\mathrm{U}(\mathcal{H})$ is a real Banach Lie group in the norm topology: Its local models are open subsets of the space $L \subset \mathrm{B}(\mathcal{H})$ of bounded skew-symmetric operators. L is a real Banach space and a real Lie algebra with respect to the commutator. The exponential map

$$\exp : L \to \mathrm{U}(\mathcal{H}), B \mapsto \exp(B) = \sum \frac{B^n}{n!} \tag{10}$$

is locally invertible and thus provides the manifold structure on the unitary group. In this way, $\mathrm{U}(\mathcal{H})$ is a Lie group with Lie algebra L.

The same procedure does not work for the strong topology (in the infinite dimensional case). Although it can be shown that the above exponential map $\exp : L \to \mathrm{U}(\mathcal{H})$ is continuous with respect to the strong topologies, it is not a local homeomorphism. A way to see that $\mathrm{U}(\mathcal{H})$ cannot be a Lie group with local models in L with respect to the strong topology was explained to me by K.-H. Neeb: Choose an orthonormal basis $(e_j)_{j\in J}$ in $\mathcal{H}$. The diagonal operators with respect to $(e_j)_{j\in J}$ and contained in $\mathrm{U}(\mathcal{H})$ form a subgroup which can be identified with the abelian group

$$K = \left\{ T = (\lambda_j)_{j\in J} : |\lambda_j| = 1 \right\} = \mathrm{U}(1)^J, \tag{11}$$

the product of infinitely many circles $\mathrm{U}(1)$. The topology on K induced from the strong topology is the product topology. Hence, K is compact. If $\mathrm{U}(\mathcal{H})$ would be a Lie group in the strong topology then K would be a Lie group as well with models in the space $D \cong \mathbb{R}^J$ of diagonal operators in L (with the product topology). However, as a compact Lie group K would have to be a finite dimensional manifold.

Note that if exp were locally invertible for the strong topologies then the same would be true for the restriction

$$\exp : D \to K, \mathbb{R}^J \to \mathrm{U}(1)^J. \tag{12}$$

But this restriction is not locally invertible, since for every strong neighbourhood $\mathcal{V} \subset K$ of $1 = \mathrm{id}_{\mathcal{H}}$ the inverse image $\exp^{-1}(\mathcal{V})$ contains all but finitely many straight lines of the form

$$\mathbb{R}_m := \left\{ T \in D : T = (\lambda_j), \lambda_j = 0 \text{ for } j \neq m, \lambda_m \in i\mathbb{R} \right\} \cong \mathbb{R}, \tag{13}$$

where $m \in \mathbb{N}$, and exp is not injective on $\mathbb{R}_m$.

According to the importance of $\mathrm{U}(\mathcal{H})$ in mathematics and physics one might be tempted to use all unitary, strongly continuous one parameter groups

$$\mathbb{R} \to \mathrm{U}(\mathcal{H}), t \mapsto B(t), t \in \mathbb{R}, \tag{14}$$

as the basic geometric and analytic information to find a manifold structure on $\mathrm{U}(\mathcal{H})$. Now, Stone's theorem states that the strongly continuous one parameter groups are exactly the one parameter groups of the following form

$$t \mapsto \exp(itA), t \in \mathbb{R}, \tag{15}$$

for self adjoint (not necessarily bounded) operators A on $\mathcal{H}$. However, the set of all self adjoint operators is not a linear space.

4. Application to Hilbert Bundles

The result of proposition 1 that $\mathrm{U}(\mathcal{H})$ with the strong topology is a topological group helps to find simpler and more transparent proofs (e.g. than those in [1] and [3]) and it gives a coherent picture when dealing with fiber bundles or with unitary representations of topological groups. In the following we exemplify the advantage of knowing that $\mathrm{U}(\mathcal{H})$ is a topological group with respect to the strong topology by applying this result to the study of Hilbert bundles. For a given topological group G the homotopy classification of all equivalence classes of principal fiber bundles over a fixed paracompact space X can be described using the classifying space $\mathrm{B}G$. (Recall that a classifying space $\mathrm{B}G$ of a topological group G is the quotient of a weakly contractible space $\mathrm{E}G$ (*i.e.* a topological space for which all its homotopy groups are trivial) by a proper free action of G. It has the property that any G-principal bundle over a paracompact space is isomorphic to a pullback of the principal bundle $\mathrm{E}G \to \mathrm{B}G$, and it is unique up to homotopy.) The significance of proposition 1 is that this can be done for $G = \mathrm{U}(\mathcal{H})$ or $\mathrm{PU}(\mathcal{H})$ with the strong topology. Let us explain the consequences for the study of Hilbert bundles:

A Hilbert bundle E over a (paracompact) space X is a locally trivial bundle $\pi : E \to X$ over X with continuous projection π such that the fibers $E_x = \pi^{-1}(x), x \in X$ are isomorphic to a separable complex Hilbert space $\mathcal{H}$ or its projectivation $\mathbb{P}(\mathcal{H})$. Here, "isomorphic" means unitarily isomorphic. In particular, this definition requires (in the case of $\mathcal{H} \cong E_x$ as the typical fiber) that there exists a cover of open subsets $V \subset X$ with bundle charts (*i.e.* homeomorphisms)

$$\phi : E|_V \to V \times \mathcal{H} \tag{16}$$

such that $\mathrm{pr}_1 \circ \phi = \pi$ and

$$\phi_x := \mathrm{pr}_2 \circ \phi|_{E_x} : E_x \to \mathcal{H} \tag{17}$$

is unitary for all $x \in X$. Thus, for $\dim \mathcal{H} = n < \infty$ the bundle E is an ordinary complex vector bundle with typical fiber $\mathbb{C}^n$ and structural group $\mathrm{U}(n)$.

The transition map for another bundle chart $\phi' : E|_{V'} \to V' \times \mathcal{H}$, $W = V \cap V' \neq \varnothing$, is

$$\phi' \circ \phi^{-1} : W \times \mathcal{H} \to W \times \mathcal{H}, \tag{18}$$

completely determined by the projection

$$\psi = \psi_{V',V} := \mathrm{pr}_2 \circ \phi' \circ \phi^{-1} : W \times \mathcal{H} \to \mathcal{H} \tag{19}$$

Now, as we have shown above in (3), ψ is continuous, if and only if the induced map

$$\hat{\psi}: W \to \mathrm{U}(\mathcal{H}), x \mapsto \left(f \mapsto \psi(x, f)\right), \tag{20}$$

is strongly continuous. $\hat{\psi}$ will not be continuous with respect the norm topology, in general. In the case of $\mathbb{P}(\mathcal{H})$ as the typical fiber of E we have analogous statements.

As a consequence, the natural principal fiber bundle $P = P_E \to X$ associated to the Hilbert bundle E (the frame bundle with fibers $P_x = \mathrm{U}(E_x, \mathcal{H})$ if $\mathcal{H}$ is the typical fiber) will be a principal fiber bundle whose structural group is $\mathrm{U}(\mathcal{H})$ with its strong topology and, in general, not with respect to the norm topology. Note that P_E will be, in addition, a principal fiber bundle with respect to the norm topology on $\mathrm{U}(\mathcal{H})$ if and only if there exists an open cover of X with bundle charts such that all the induced transition maps $\hat{\psi}: W \to \mathrm{U}(\mathcal{H})$ are norm continuous. Let us call such a bundle "norm-defined".

In the case that $\mathbb{P}(\mathcal{H})$ is the typical fiber of E (we call such bundles projective Hilbert bundles) we have analogous results for the associated principal bundle P_E (with fibers $P_x = \mathrm{U}(E_x, \mathbb{P}\mathcal{H})$: The structural group is $\mathrm{PU}(\mathcal{H})$ with the strong topology in general. Moreover, whenever E is norm-defined PE can also be viewed as to be a principal fiber bundle with structural group the projective unitary group $\mathrm{PU}(\mathcal{H})$ in its norm topology.

In order to classify the Hilbert bundles over X it is enough to classify the principal fiber bundles with structural groups $\mathrm{U}(\mathcal{H})$ resp. $\mathrm{PU}(\mathcal{H})$. Let $Princ^N_{\mathrm{U}(\mathcal{H})}(X)$ the set of isomorphism classes of principal fiber bundles with $\mathrm{U}(\mathcal{H})$ in the norm topology and correspondingly $Princ^S_{\mathrm{U}(\mathcal{H})}(X)$ the set of isomorphism classes of principal fiber bundles with $\mathrm{U}(\mathcal{H})$ in the strong topology. Analogously, we define $Princ^A_{\mathrm{PU}(\mathcal{H})}(X)$ for $A \in \{N, S\}$.

Unitary group (vector bundles): Since the unitary group is contractible in both topologies every principal bundle P over X is trivial:

$$Princ^A_{\mathrm{U}(\mathcal{H})}(X) \cong \left[X, \mathrm{BU}(\mathcal{H})^A\right] \cong \left\{\left[X \times \mathrm{U}(\mathcal{H})\right]\right\} \tag{21}$$

for $A \in \{N, S\}$. (Here, $[X, Y]$ denotes the set of homotopy classes of continuous maps between topological spaces X and Y, and $[P]$ denotes the equivalence class of the principal bundle P over X.) For an arbitrary Hilbert bundle with typical fiber $\mathcal{H}$ this implies that it is already isomorphic to the trivial bundle $X \times \mathcal{H}$ For the norm-defined bundles E the associated principal bundle P_E is in $Princ^A_{\mathrm{U}(\mathcal{H})}(X)$ and an isomorphism $E \cong X \times \mathcal{H}$ can be found which is locally given by transition functions which are induced by norm continuous $W \to \mathrm{U}(\mathcal{H})$. Note, that the classifying spaces $\mathrm{BU}(\mathcal{H})^A$ are weakly contractible for $A \in \{N, S\}$.

Projective unitary group (projective bundles): We know already that $\mathrm{PU}(\mathcal{H})$ is a $K(\mathbb{Z}, 2)$ for both topologies on the projective unitary group which we will indicate by a superscript A. From the homotopy sequence corresponding to the universal bundle

$$\mathrm{PU}(\mathcal{H})^A \to \mathrm{EPU}(\mathcal{H})^A \to \mathrm{BPU}(\mathcal{H})^A \tag{22}$$

one concludes that $\mathrm{BPU}(\mathcal{H})^A$ is an Eilenberg-MacLane space $K(\mathbb{Z},3)$. Now, the homotopy classification of principal fiber bundles asserts that there is a bijection between $Princ^A_{\mathrm{PU}(\mathcal{H})}(X)$ and $\left[X,\mathrm{BPU}(\mathcal{H})^A\right]$, the set of homotopy classes of continuous $X \to \mathrm{BPU}(\mathcal{H})^A$. For a $K(\mathbb{Z},3)$ this is cohomology: $\left[X,\mathrm{BPU}(\mathcal{H})^A\right] \cong H^3(X,\mathbb{Z})$. We arrive at the following result which is essentially contained in a different form in [3]:

Proposition 4:

- *The isomorphism classes of projective Hilbert bundles over X are in one-to-one correspondence to* $H^3(X,\mathbb{Z}) \cong \left[X,\mathrm{BPU}(\mathcal{H})^S\right] \cong Princ^S_{\mathrm{PU}(\mathcal{H})}(X)$.
- *The isomorphism classes of norm-defined projective Hilbert bundles over X are also in one-to-one correspondence to* $H^3(X,\mathbb{Z}) \cong \left[X,\mathrm{BPU}(\mathcal{H})^N\right] \cong Princ^N_{\mathrm{PU}(\mathcal{H})}(X)$ *where the isomorphisms of the Hilbert bundles are given by norm continuous transition maps.*

Note, that the zero element of $H^3(X,\mathbb{Z})$ represents the class of all trivial bundles which also can be described as the class of projective Hilbert bundles E of the form $\mathbb{P}F$ where F is a true vector bundle with fibers $F_x \cong \mathcal{H}$.

5. Conclusion

The property of $\mathrm{U}(\mathcal{H})$ being a topological group serves as a basis for further research in various areas in mathematics and physics where $\mathrm{U}(\mathcal{H})$ is a symmetry group. Such a research will be supported by that fact, that $\mathrm{U}(\mathcal{H})$ is, in addition, contractible and metrizable if $\mathcal{H}$ is infinite dimensional and separable. This has been exemplified in the last part of this paper by deducing the classification of Hilbert bundles from these results concerning the strong topology on $\mathrm{U}(\mathcal{H})$.

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