From London to Hanoi and back — graphs for neuropsychology

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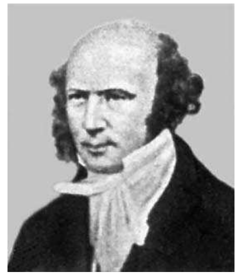
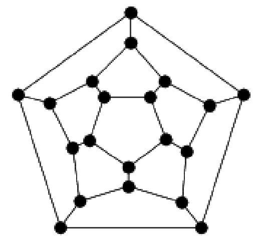
Icosian Game (1859)

N. L. Biggs, E. K. Lloyd, R. J. Wilson, Graph Theory 1736–1936, Clarendon Press, Oxford, 1986.

Icosian Game



W. R. Hamilton

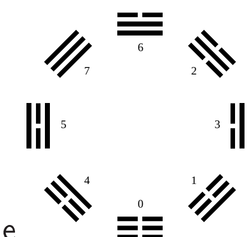


hamiltonian cycle (path) $C_{|G|} \subset G$ ($P_{|G|} \subset G$) in a dodecahedral graph

Chinese Rings and Gray Code

S. N. Afriat, The Ring of Linked Rings, Duckworth, London, 1982.

Chinese Rings



modeled by state graph $B^n := \{0,1\}^n \sim P_{2^n}$

leads to Gros sequence $1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, \ldots$ and Gray code

hamiltonian cycle on n-cube

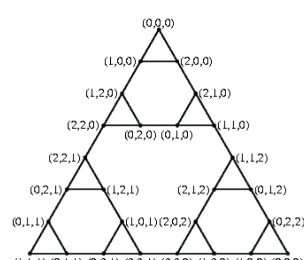
Tower of Hanoi (1883)

A. M. Hinz, The Tower of Hanoi, Enseign. Math. (2) 35 (1989), 289-321.

A. M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, European J. Combin. 26 (2005), 693–708.

Title plate

F. É. A. Lucas



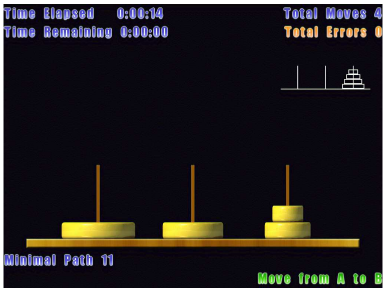
modeled by state graph $T^n := \{0,1,2\}^n$

connected, planar, hamiltonian, $\operatorname{diam}(T^n) = 2^n - 1$, $\chi(T^n) = 3 = \chi'(T^n)$

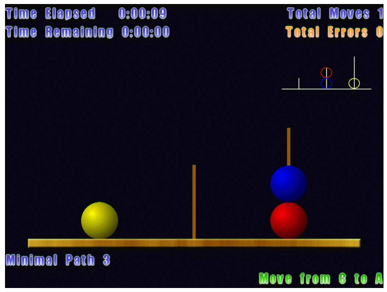
at most two shortest paths between any two vertices

Wiener index $W(T^n) = \frac{233}{885} 18^n - \frac{1}{6} 9^n + \frac{3}{59}\left(2 + \frac{3}{17}\sqrt{17}\right)\left(\frac{1}{2}\left(5 + \sqrt{17}\right)\right)^n - \frac{3}{10} 3^n + \frac{3}{59}\left(2 - \frac{3}{17}\sqrt{17}\right)\left(\frac{1}{2}\left(5 - \sqrt{17}\right)\right)^n$

psychological test tool for the Tower of Hanoi and Tower of London (1982) tasks



Tower of Hanoi platform



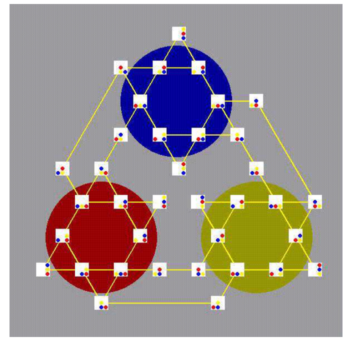
Tower of London platform

T. Shallice, Specific impairments of planning, Philos. Trans. Roy. Soc. London B 298 (1982), 199–209.

A. M. Hinz et al., A Mathematical Model and Computer Tool for the Tower of Hanoi and Tower of London Tasks, in preparation, 2007.

Tower of London modeled by state graph L_3^3: connected, planar, non-hamiltonian, $\mathrm{diam}(L_3^3) = 8$

up to eight shortest paths between two vertices



subgraphs of Oxford graphs O_3^n: connected, non-planar $(n \geq 3)$

digraph for Lucas's Second Problem (1883)

Comparison of Tower Graphs

T^n, O_3^n, L_3^3 with its variants $L_3^2 \subset O_3^2$, $\widetilde{L}_3^3 \subset O_3^3$, $\widetilde{L}_3^4 \subset O_3^4$, $\widetilde{L}_3^5 \subset O_3^5$, and the 4-pegs versions $Q^n := \{0,1,2,3\}^n$, O_4^n, $L_4^4 \subset O_4^4$, and $L_4^5 \subset O_4^5$ have the following quantitative properties:

G	$\lvert G \rvert$	$\lVert G \rVert$	δ	$\overline{\deg}$	Δ	$\overline{d}$	$\mathrm{diam}(G)$	equi-sets (number)	equi-sets (sizes)
T^2	9	12	2	2.667	3	1.778	3	12	12 × 6
T^3	27	39	2	2.889	3	3.893	7	117	117 × 6
T^4	81	120	2	2.963	3	8.102	15	1080	1080 × 6
T^5	243	363	2	2.988	3	16.523	31	9801	9801 × 6
T^6	729	1092	2	2.996	3	33.370	63	88452	88452 × 6
O_3^2	12	18	2	3.000	4	1.958	4	12	10 × 12 + 2 × 6
O_3^3	60	108	2	3.600	6	3.647	7	102	95 × 36 + 6 × 18 + 1 × 12
O_3^4	360	720	2	4.000	6	5.596	10	914	881 × 144 + 33 × 72
O_3^5	2520	5400	2	4.286	6	7.738	13	8885	8748 × 720 + 137 × 360
L_3^2	10	14	2	2.800	4	1.860	4	23	22 × 4 + 1 × 2
L_3^3	36	54	2	3.000	4	4.236	8	210	210 × 6
$\widetilde{L}_3^3$	54	96	2	3.778	6	3.562	7	239	238 × 12 + 1 × 6
$\widetilde{L}_3^4$	264	504	2	3.818	6	5.828	10	2893	2893 × 24
$\widetilde{L}_3^5$	1320	2520	2	3.818	6	8.868	14	14509	14509 × 120
Q^2	16	36	3	4.500	5	1.781	3	13	7 × 24 + 6 × 12
Q^3	64	168	3	5.250	6	3.035	5	182	154 × 24 + 28 × 12
Q^4	256	720	3	5.625	6	4.666	9	2780	2660 × 24 + 120 × 12
Q^5	1024	2976	3	5.813	6	6.733	13	43896	43400 × 24 + 496 × 12
O_4^2	20	48	3	4.800	6	1.910	4	13	4 × 48 + 7 × 24 + 1 × 12 + 1 × 8
O_4^3	120	360	3	6.000	9	3.290	6	126	76 × 144 + 44 × 72 + 1 × 48 + 5 × 24
O_4^4	840	2880	3	6.857	12	4.815	9	1400	1068 × 576 +298 × 288 + 1 × 192 + 9 × 144 + 23 × 96 + 1 × 72
O_4^5	6720	25200	3	7.500	12	6.452	12	16852	14584 × 2880 + 2146 × 1440 + 2 × 960 + 120 × 480
L_4^4	480	1464	3	6.100	9	4.765	9	9580	9580 × 24
L_4^5	2640	8280	4	6.273	9	6.819	12	58058	58058 × 120

Higher Base Hanoi Graphs

presumed **m**inimal solution for perfect to perfect problem in Q^n

Frame-Stewart-Conjecture: $pms = d(p, \widetilde{p})$; true for $n \leq 30$.

eccentricity $\varepsilon(p) = d(p, \widetilde{p})$ for $n < 15$, but:

n	$d(p,\widetilde{p})$	$\varepsilon(p)$
15	129	130
20	289	294
21	321	341
22	385	394

$\mathrm{diam}(Q^n) = d(p, \widetilde{p})$ for $n \leq 12$; open: $\mathrm{diam}(Q^n) = \varepsilon(p)$?

$H_p^n \subset O_p^n$ Hanoi/Oxford graphs of base p and exponent n; open: diameters

$\chi(H_p^n) = p$, $\chi'(H_p^n) = \Delta(H_p^n) = \binom{p}{2} - \binom{p-n}{2} \cdot (n \leq p-2)$ for $p, n > 1$

A.M.Hinz & D.Parisse (2006)

H_p^n hamiltonian; planar only for $p < 4$ and $p = 4$, $n = 1, 2$

connections to Sierpiński graphs S_p^n: $S_2^n \sim B^n$, $S_3^n \sim T^n$, but S_p^n and H_p^n are not isomorphic for $p > 3, n > 1$.

J.-B. Bode, A. M. Hinz, Results and open problems on the Tower of Hanoi, Congr. Numer. 139 (1999), 113–122.

R. E. Korf, A. Felner, Heuristic Search for Multiple Goal States, Preprint, 2006.

A. M. Hinz, D. Parisse, On the Planarity of Hanoi Graphs, Exposition. Math. 20 (2002), 263–268.

S. Klavžar, U. Milutinović, Graphs $S(n,k)$ and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47(122) (1997), 95–104.