SCALING OF THE GROUND-STATE ENERGY OF RELATIVISTIC IONS IN HIGH LOCALLY BOUNDED MAGNETIC FIELDS

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Abstract. We consider the pseudorelativistic Chandrasekhar/Herbst operator h^H for the description of relativistic one-electron ions in a locally bounded magnetic field. We show that for Coulomb potentials of strength $\gamma < 2/\pi$, the spectrum of h^H is discrete below m (the electron mass). For magnetic fields in the class $\mathbf{B}_A(\mathbf{x}) = B \cdot \frac{1+\tau}{2}(|x_1|^\tau + |x_2|^\tau)\,\mathbf{e}_z$, the ground-state energy of h^H decreases according to $B^{1/(2+\tau)}$ as $B \to \infty$ for $0 \le \tau < \tau_c$ where τ_c is some critical value depending on γ.

1. Introduction

Relativistic one-electron ions are described by the Dirac operator. In the presence of a Coulomb field V and a magnetic field $\mathbf{B}_A$ generated by a vector potential $\mathbf{A}$ this operator is given by (in relativistic units, $\hbar = c = 1$)

$$D_A \;=\; \boldsymbol{\alpha}\,(\mathbf{p} - e\mathbf{A}) \;+\; \beta\, m, \qquad V \;=\; -\frac{\gamma}{x}. \tag{1.1}$$

The particle mass is denoted by m, $\boldsymbol{\alpha}$ and β are the Dirac matrices, $\mathbf{p} = -i\boldsymbol{\nabla}$ the momentum operator, $x = |\mathbf{x}|$ and $\gamma = Ze^2$ the electric potential strength (Z is the charge of the nucleus which is fixed at the origin and $e^2 \approx 1/137.04$ is the fine structure constant).

In order to avoid dealing with the negative continuum which, in contrast to the nonrelativistic case, causes the Dirac operator to be unbounded from below, semibounded pseudorelativistic operators are often introduced. One way to obtain such a semibounded operator is the restriction of the Dirac operator to the positive spectral subspace of the free Dirac operator. When a vector potential is present one has to include $\mathbf{A}$ in the projector onto the positive spectral subspace in order to retain gauge invariance [14]. As a result one obtains the Brown-Ravenhall operator [3, 14, 9], acting in the Hilbert space $L_2(\mathbb{R}^3) \otimes \mathbb{C}^2$ [5],

$$h^{BR} \;=\; E_A \;-\; \gamma A_E \left(\frac{1}{x} \;+\; \frac{\boldsymbol{\sigma}\mathbf{p}_A}{E_A + m}\,\frac{1}{x}\,\frac{\boldsymbol{\sigma}\mathbf{p}_A}{E_A + m} \right) A_E \tag{1.2}$$

with $A_E = \left(\frac{E_A+m}{2E_A}\right)^{1/2}$. The kinetic energy operator is given by

$$|D_A| =: E_A = \sqrt{(\boldsymbol{\sigma}(\mathbf{p}-e\mathbf{A}))^2+m^2} \geq m. \tag{1.3}$$

Abbreviating $\mathbf{p}-e\mathbf{A} = \mathbf{p}_A$ one can rewrite $(\boldsymbol{\sigma}\mathbf{p}_A)^2 = (\mathbf{p}_A)^2 - e\boldsymbol{\sigma}\mathbf{B}_A$ where $\mathbf{B}_A = \boldsymbol{\nabla}\times\mathbf{A}$ and $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices. For $\mathbf{A}\in L_{2,loc}(\mathbb{R}^3)$ the form domain $Q(E_A) = \mathcal{D}(E_A^{\frac{1}{2}})$ is the Sobolev space $H_{\frac{1}{2}}(\mathbb{R}^3)\otimes\mathbb{C}^2$.

It was shown in [17] that h^{BR} is bounded from below if $\mathbf{A}\in L_{loc}^\infty(\mathbb{R}^3)$ and $\gamma < \gamma_{BR} := 2/(\frac{\pi}{2}+\frac{2}{\pi})$. In the case of a constant magnetic field $\mathbf{B}_A = B\mathbf{e}_z$ it could be proven that the lower bound of h^{BR} decreases proportional to $B^{\frac{1}{2}}$ and that the ground-state energy behaves like $-cB^{\frac{1}{2}}$ for $B\to\infty$ if $\gamma < \frac{2}{\pi}$ [11]. The advantage of a constant magnetic field is the explicit knowledge of the eigenstates of the Pauli operator $(\boldsymbol{\sigma}\mathbf{p}_A)^2$ [20, p.196], respectively of E_A. Since the potential part of h^{BR} depends nontrivially on E_A, such eigenstates are crucial for the construction of trial functions which are used to determine an upper bound for the ground-state energy of h^{BR}.

It is an open question how the asymptotic dependence of the ground-state energy on the magnetic field will change if locally bounded, but nonconstant magnetic fields are considered. For such fields no exact eigenstates of the Pauli operator are known, and the nonlocality of E_A prohibits variational calculations.

Therefore we consider instead of h^{BR} the pseudorelativistic Herbst operator (also termed Chandrasekhar operator [2, 8]) which has a simpler structure and also acts in the Hilbert space $L_2(\mathbb{R}^3)\otimes\mathbb{C}^2$,

$$h^H = E_A + V. \tag{1.4}$$

It retains the relativistic kinematics, but omits the E_A-dependent factors, arising from the projectors, in the potential. There are seminal works using the Herbst operator for the determination of stability and spectral properties of multiparticle systems (e.g. [16, 13]). Although giving less accurate estimates for the relativistic atomic binding energies than h^{BR}, it is shown below that h^H has the same asymptotic B-dependence as h^{BR} for a constant field. So we conjecture that our results for locally bounded fields will hold in the case of related relativistic operators as well.

The aim of the present work is to show the semiboundedness of h^H, the relative form boundedness of V and the localization of the essential spectrum of h^H for all locally bounded magnetic fields and $\gamma < 2/\pi$ (Theorem 1; proof in Section 3). Moreover, for a special class of unbounded magnetic fields, introduced in Section 2, it is proven that a bound ground state of h^H exists for subcritical growth of $\mathbf{B}_A(\mathbf{x})$ as $x\to\infty$, and that the ground-state energy scales asymptotically with the magnetic field strength according to a power law (Theorem 2). This is based on a scaling property of the Herbst operator which is derived in Section 4. The existence of a discrete ground state is investigated by means of a variational calculation (Section 5) and it is shown that it depends on the interplay between the growth of $\mathbf{B}_A$ and the strength of V. The proof of Theorem 2 is completed in Section 6.

2. Main results for the Herbst operator

It is well known from the field-free case ($\mathbf{A}=0$) that the Herbst operator is semibounded for $\gamma \leq \frac{2}{\pi}$ [8]. For locally bounded magnetic fields we have a similar result.

Theorem 1. *Let $h^H = E_A + V$ be the Herbst operator for an electron in a magnetic field $\mathbf{B}_A \in L^\infty_{loc}(\mathbb{R}^3)$ and in a central Coulomb field of strength $\gamma < 2/\pi$ $(Z \leq 87)$. Then for all $m \geq 0$*

(i) *h^H is bounded from below in the form sense on $H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^2$.*

(ii) *V is form bounded with respect to E_A with form bound smaller than one. Hence, by the KLMN theorem [19, p.167], h^H can be extended to a self-adjoint operator with form domain $Q(E_A)$.*

(iii) *For the essential spectrum of h^H one has*

$$\sigma_{ess}(h^H) \; = \; \sigma_{ess}(E_A) \; \subset \; [m, \infty). \tag{2.1}$$

As a consequence, all eigenstates of h^H below m, if existing, are discrete.

Let us now consider the special class of locally bounded magnetic fields originating from the vector potentials

$$\mathbf{A}(\mathbf{x}) \; = \; \frac{B}{2} \, (-x_2 \, |x_2|^\tau, x_1 \, |x_1|^\tau, 0), \qquad \tau > 0. \tag{2.2}$$

$\mathbf{A}$ obeys $\boldsymbol{\nabla}\mathbf{A} = 0$ (which simplifies the evaluation of $(\mathbf{p} - e\mathbf{A})^2$) and generates the field $\mathbf{B}_A = \boldsymbol{\nabla} \times \mathbf{A}$,

$$\mathbf{B}_A(\mathbf{x}) \; = \; B \, \frac{1+\tau}{2} \, (0, 0, |x_1|^\tau + |x_2|^\tau) \; \in L^\infty_{loc}(\mathbb{R}^3). \tag{2.3}$$

For the limiting case $\tau = 0$ we obtain the constant magnetic field $\mathbf{B}_A = (0, 0, B)$.

Theorem 2. *Let $h^H = E_A + V$ be the Herbst operator for an electron in a magnetic field $\mathbf{B}_A$ of the class (2.3) and in a Coulomb field of strength $0.1 \lesssim \gamma < \frac{2}{\pi}$.*

(i) *For every field strength $B > 2Z^2 B_0$ there exists a critical field growth $\tau_c(\gamma, B) > 0$ such that h^H has a discrete ground state for every $\tau < \tau_c$. ($B_0 = 2.35 \times 10^9$ G is the unit field.)*

(ii) *For $B \to \infty$ and for $\tau < \tau_c(\gamma)$, the ground-state energy E_g of h^H behaves like*

$$E_g \; \sim \; -c \, B^{\frac{1}{2+\tau}} \tag{2.4}$$

where $c > 0$ is some constant.

3. General properties of the Herbst operator: proof of Theorem 1

3.1. Semiboundedness of h^H.

When we represent a locally bounded field $\mathbf{B}_A$ by the vector potential [7]

$$\mathbf{A}_{x_0}(\mathbf{x}) \;=\; \int_0^1 t\,dt\,\mathbf{B}_A(\mathbf{x}_0+t(\mathbf{x}-\mathbf{x}_0))\wedge(\mathbf{x}-\mathbf{x}_0) \tag{3.1}$$

which satisfies $\boldsymbol{\nabla}\times\mathbf{A}_{x_0}=\mathbf{B}_A$ for arbitrary $\mathbf{x}_0\in\mathbb{R}^3$, it is obvious that $\mathbf{A}_{x_0}$ is also locally bounded. Thus without restriction we can assume $\mathbf{A}\in L^\infty_{loc}(\mathbb{R}^3)$ if $\mathbf{B}_A\in L^\infty_{loc}(\mathbb{R}^3)$.

Following the ideas of [17] in the proof of the semiboundedness of the Brown-Ravenhall operator for $\gamma<\gamma_{BR}$, we introduce smooth functions $\chi_1,\chi_2\in C^\infty(\mathbb{R}^3)$ mapping to $[0,1]$ by

$$\chi_2(\mathbf{x}) := \begin{cases} 0, & x<R_0 \\ 1, & x\geq R_1 \end{cases}, \qquad\qquad \chi_1^2+\chi_2^2 \;=\; 1, \tag{3.2}$$

where $R_0<R_1$ are some positive real numbers.

For the localization of the kinetic energy operator E_A we use an estimate by Lenzmann and Lewin [12, Proof of Lemma A.2],

$$\frac{1}{2}\,(E_A\chi_k^2+\chi_k^2E_A)\;\geq\;\chi_k\,E_A\,\chi_k\;+\;\frac{1}{2\pi}\,I_k \tag{3.3}$$

$$I_k := \int_0^\infty \frac{1}{s+E_A^2}\,[\,[\,E_A^2,\chi_k],\;\chi_k]\,\frac{1}{s+E_A^2}\,\sqrt{s}\,ds,$$

which is based on the identity,

$$E_A \;=\; \frac{1}{\pi}\int_0^\infty \frac{E_A^2}{s+E_A^2}\,\frac{ds}{\sqrt{s}}. \tag{3.4}$$

Both relations, (3.3) and (3.4), in fact hold for arbitrary operators E_A and χ_k. Performing in (3.3) the sum over $k=1,2$ and using (3.2) we obtain for $\psi\in H_{1/2}(\mathbb{R}^3)\otimes\mathbb{C}^2$,

$$(\psi,\,E_A\,\psi)\;\geq\;\sum_{k=1}^{2}(\chi_k\psi,\,E_A\,\chi_k\psi)\;+\;\frac{1}{2\pi}\,(\psi,\,(I_1+I_2)\,\psi). \tag{3.5}$$

In order to show the boundedness of the integrals $I_k,\;k=1,2,$ we note that

$$[E_A^2,\chi_k]\;=\;[\,(\mathbf{p}-e\mathbf{A})^2-e\boldsymbol{\sigma}\mathbf{B}_A+m^2,\,\chi_k]\;=\;[p^2,\chi_k]\;+\;2i\,e\mathbf{A}\,(\boldsymbol{\nabla}\chi_k). \tag{3.6}$$

The last term in (3.6) is a bounded function since $\mathbf{A}\in L^\infty_{loc}(\mathbb{R}^3)$ and $\boldsymbol{\nabla}\chi_k\in C_0^\infty(\mathbb{R}^3)$. It follows that $[\,[\,E_A^2,\chi_k],\,\chi_k]\;=\;[\,[p^2,\chi_k],\,\chi_k]\;=\;-2(\boldsymbol{\nabla}\chi_k)^2$ which is bounded.

Consider for $m>0$,

$$\left\|\frac{1}{s+E_A^2}\,(\boldsymbol{\nabla}\chi_k)^2\,\frac{1}{s+E_A^2}\right\|\;=\;\sup_{\|\varphi\|=\|\psi\|=1}\left|(\psi,\,\frac{1}{s+E_A^2}\,|\boldsymbol{\nabla}\chi_k|\cdot|\boldsymbol{\nabla}\chi_k|\,\frac{1}{s+E_A^2}\,\varphi)\right|$$

$$\leq \sup_{\|\varphi\|=\|\psi\|=1} \left\| |\nabla\chi_k| \frac{1}{s+E_A^2} \psi \right\| \cdot \left\| |\nabla\chi_k| \frac{1}{s+E_A^2} \varphi \right\| \leq c_k^2 \frac{1}{(s+m^2)^2}, \quad (3.7)$$

where we have used that $c_k := \max_{x \leq R_1} |\nabla\chi_k|$ and $E_A \geq m$. Thus the integrand operator in I_k has an integrable bound which renders I_k finite, $k = 1, 2$.

In (3.5) we use $E_A \geq 0$ to estimate $(\chi_2\psi, E_A\,\chi_2\psi) \geq 0$. This leads to

$$(\psi, E_A\,\psi) \geq (\chi_1\psi, E_A\,\chi_1\psi) - L\,(\psi,\psi) \quad (3.8)$$

where $L > 0$ is the bound of $\frac{1}{2\pi}(I_1 + I_2)$.

For the potential we have the decomposition

$$(\psi, V\,\psi) = (\psi, \chi_1\,V\,\chi_1\,\psi) + (\psi, \chi_2\,V\,\chi_2\,\psi). \quad (3.9)$$

Using that V is bounded except near $x = 0$ where χ_2 vanishes, we can estimate

$$|(\psi, \chi_2\,V\,\chi_2\,\psi)| \leq \sup_{x\in\mathbb{R}^3} (\chi_2\,V\,\chi_2) \cdot \|\psi\|^2 =: C_V\,\|\psi\|^2 \quad (3.10)$$

where C_V is some finite constant.

Moreover, since $\chi_1 \in C_0^\infty(\mathbb{R}^3)$ such that $\mathbf{B}_A\chi_1$ is bounded, $\chi_1\,V\,\chi_1$ can be estimated by $\chi_1\,E_A\chi_1$ in the form sense like in the case of a constant magnetic field. This is done with the help of the diamagnetic inequality (see e.g. [1]),

$$|(\chi_1\,\psi, -\frac{\gamma}{x}\,\chi_1\,\psi)| \leq \frac{\gamma\pi}{2}\,(\chi_1\,\psi, \sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2}\,\chi_1\,\psi). \quad (3.11)$$

We use $(\mathbf{p}-e\mathbf{A})^2 + m^2 = E_A^2 + e\boldsymbol{\sigma}\mathbf{B}_A$ and have the estimate

$$(\chi_1\psi, (E_A^2 + e\boldsymbol{\sigma}\mathbf{B}_A)\,\chi_1\psi) \leq (\chi_1\psi, E_A^2\,\chi_1\psi) + \max_{x\leq R_1} |e\mathbf{B}_A|\,(\chi_1\psi, \chi_1\psi). \quad (3.12)$$

A consequence is the relative form boundedness,

$$(\chi_1\psi, \sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2}\,\chi_1\psi) \leq (\chi_1\psi, E_A\,\chi_1\psi) + \tilde{C}\,(\chi_1\psi, \chi_1\psi), \quad (3.13)$$

where $\tilde{C} > 0$ is some constant. With $\|\chi_1\psi\|^2 \leq \|\psi\|^2$ this leads to the estimate

$$(\psi, h^H\,\psi) \geq (1 - \frac{\gamma\pi}{2})\,(\chi_1\,\psi, E_A\,\chi_1\,\psi) - (L + C_V + C_B)\,\|\psi\|^2 \quad (3.14)$$

where we have abbreviated $C_B := \frac{\gamma\pi}{2}\,\tilde{C}$. Note that for C_B to be finite, the form bound in (3.13) has to be infinitesimally larger than one [19, p.169]. Thus for subcritical potential strength, $\gamma < 2/\pi$, h^H is form bounded from below, which proves Theorem 1(i) for $m \neq 0$.

In order to cover the case $m = 0$ we introduce the operator $h_-^H := h^H - m$ and note that when h^H is bounded from below, this is also true for h_-^H.

The m-dependence of h^H is the one of E_A. We have from (1.3)

$$E_A - m = \frac{E_A^2 - m^2}{E_A + m} = \frac{(\boldsymbol{\sigma}(\mathbf{p}-e\mathbf{A}))^2}{E_A + m} \quad (3.15)$$

which is monotonically decreasing with m. Moreover, $E_A - m$ is continuous in $m \geq 0$. In fact for $\psi \in \mathcal{D}(E_A^{\frac{1}{2}})$, indicating explicitly the m-dependence of the operators,

$$|(\psi, (E_A(m) - E_A(m_0))\,\psi)| \;=\; \left|(\psi,\; \frac{m^2 - m_0^2}{E_A(m) + E_A(m_0)}\;\psi)\right|$$
$$\le\; |m - m_0|\;\|\psi\|^2. \tag{3.16}$$

Let $-C_m := -(L + C_V + C_B)$ be the lower bound of h^H at a given $m > 0$. Then

$$(\psi, h^H(m=0)\,\psi) \;\ge\; (\psi, (h^H(m) - m)\,\psi)$$
$$\ge\; (1 - \frac{\gamma\pi}{2})\;(\chi_1\psi, E_A(m)\,\chi_1\,\psi)\;\; - (C_m + m)\;\|\psi\|^2 \tag{3.17}$$

which shows the boundedness of h^H from below when $m = 0$.

3.2. **Relative form boundedness of the potential.**

Without restriction we can assume $m \neq 0$ (if $m = 0$ we can use the monotonicity of E_A which gives $(\psi, E_A(m)\,\psi)\; \le (\psi, E_A(0)\,\psi) + m(\psi,\psi)$ for any fixed m). With (3.9) – (3.13) we have

$$|(\psi, V\;\psi)| \;\le\; \frac{\gamma\pi}{2}\;(\chi_1\psi, E_A\chi_1\,\psi) \;+\; (C_B + C_V)\;\|\psi\|^2. \tag{3.18}$$

Using (3.8) we can estimate further

$$|(\psi, V\;\psi)| \;\le\; \frac{\gamma\pi}{2}\;(\psi, E_A\;\psi) \;+\; (\frac{\gamma\pi}{2}\;L + C_B + C_V)\;\|\psi\|^2. \tag{3.19}$$

3.3. **Discreteness of the spectrum of h^H below m.**

With $E_A \ge m$ we have $\sigma(E_A) \subset [m,\infty)$ and hence $\sigma_{ess}(E_A) \subset [m,\infty)$.

We note that for a suitable $\mu > 0$, $h^H + \mu > 0$ from Theorem 1(i). We will show that the difference R_μ of the resolvents of h^H and E_A is compact. Following the corresponding proof for the Brown-Ravenhall operator [9] we decompose

$$R_\mu := \;\frac{1}{h^H + \mu} \;-\; \frac{1}{E_A + \mu} \;=\; -\left\{\frac{1}{E_A + \mu}\;V\;\frac{1}{(E_A + \mu)^{\frac{1}{2}}}\right\}\;\left[(E_A + \mu)^{\frac{1}{2}}\;\frac{1}{h^H + \mu}\right] \tag{3.20}$$

and establish compactness of the operator in curly brackets which we write in the following way,

$$-\;\frac{1}{E_A + \mu}\;V\;\frac{1}{(E_A + \mu)^{\frac{1}{2}}} \;=\; \gamma\left\{\frac{1}{E_A + \mu}\;\frac{1}{x^{\frac{1}{2}}}\right\}\;\left[\frac{1}{x^{\frac{1}{2}}}\;\frac{1}{(E_A + \mu)^{\frac{1}{2}}}\right] \tag{3.21}$$

The operator $x^{-\frac{1}{2}}(E_A + \mu)^{-1}$ is compact (see remark to [9, Lemma 2] and the proof of [11, Lemma 1]), and hence also its adjoint. The operator in square brackets is bounded since, from (3.19) with $\tilde{C}_m := \frac{\pi}{2}\,L + (C_B + C_V)/\gamma$,

$$\|\frac{1}{x^{\frac{1}{2}}}\;\frac{1}{(E_A + \mu)^{\frac{1}{2}}}\;\psi\|^2 \;\le\; \frac{\pi}{2}\;(\frac{1}{(E_A + \mu)^{\frac{1}{2}}}\;\psi, E_A\;\frac{1}{(E_A + \mu)^{\frac{1}{2}}}\;\psi)$$
$$+\;\tilde{C}_m\;\|\frac{1}{(E_A + \mu)^{\frac{1}{2}}}\;\psi\|^2 \;\le\; \left(\frac{\pi}{2} \;+\; \tilde{C}_m\;\|\frac{1}{(E_A + \mu)}\|\right)\;\|\psi\|^2. \tag{3.22}$$

The boundedness of the second factor in (3.20), $(E_A+\mu)^{\frac{1}{2}}\,(h^H+\mu)^{-\frac{1}{2}}\cdot T_\mu$ with $T_\mu := (h^H+\mu)^{-\frac{1}{2}}$ bounded, follows also from the relative form boundedness of V. In fact we have, with a constant $c>0$ to be determined later and $\varphi \in \mathcal{D}(E_A^{\frac{1}{2}})$,

$$\begin{aligned} c\,(\varphi,(h^H+\mu)\,\varphi) \;&\geq\; c\,(\varphi,(E_A+\mu)\,\varphi) \;-\; c\,|(\varphi, V\,\varphi)| \\ &\geq\; c\,(\varphi,(E_A+\mu)\,\varphi) \;-c\,\frac{\gamma\pi}{2}\,(\varphi, E_A\,\varphi) \;-c\gamma\,\tilde{C}_m\,\|\varphi\|^2. \end{aligned} \tag{3.23}$$

If we choose $c \geq \frac{1}{1-\gamma\pi/2}$ and $\mu \geq \frac{2\tilde{C}_m}{\pi}$ then the rhs of (3.23) is $\geq\,(\varphi,(E_A+\mu)\,\varphi)$. Setting $\varphi := (h^H+\mu)^{-\frac{1}{2}}\,\psi$ we derive the desired boundedness,

$$c\,\|\psi\|^2 \;\geq\; \|(E_A+\mu)^{\frac{1}{2}}\,\frac{1}{(h^H+\mu)^{\frac{1}{2}}}\,\psi\|^2. \tag{3.24}$$

As a result, R_μ is compact and consequently, $\sigma_{ess}(h^H)\;=\sigma_{ess}(E_A)$.

4. Scaling property

Let us restrict ourselves to the class of locally bounded magnetic fields (2.3) originating from the vector potentials (2.2).

In order to derive the scaling property of the Herbst operator which provides an easy way for treating an arbitrarily large magnetic field strength, we set $B=\mu_0 B_0$ with a dimensionless scaling parameter μ_0. Then with B_0 the unit field from Theorem 2 and $\mu_0\to\infty$ we have $B\to\infty$. Define the scaled coordinates $\tilde{x}_k := \mu_0^\delta x_k$ and, correspondingly, the scaled momenta $\tilde{p}_k := \mu_0^{-\delta} p_k,\;\; k=1,2,3,$ with $\delta>0$ yet to be determined. Then with $\mu_0>0$,

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{B_0}{2}\,\mu_0^{1-\delta(1+\tau)}\,(-\tilde{x}_2\,|\tilde{x}_2|^\tau, \tilde{x}_1\,|\tilde{x}_1|^\tau, 0) \;=:\mu_0^{1-\delta(1+\tau)}\mathbf{A}_0(\tilde{\mathbf{x}})$$

$$E_A^2 \;=\left[\sum_{k=1}^{2}\left(\tilde{p}_k - e\mu_0^{1-\delta(2+\tau)} A_{0k}(\tilde{\mathbf{x}})\right)^2 -\; e\sigma_3\,\frac{1+\tau}{2}\,B_0\mu_0^{1-\delta(2+\tau)}\,(\,|\tilde{x}_1|^\tau + |\tilde{x}_2|^\tau)\right] \cdot\mu_0^{2\delta}\;+\;p_3^2+m^2. \tag{4.1}$$

A necessary condition for the scaling property to hold is $1-\delta(2+\tau)=0$. It is satisfied for $\delta=\frac{1}{2+\tau}$ and we get

$$E_A \;=\; \mu_0^\delta\,\tilde{E}_A, \tag{4.2}$$

$$\tilde{E}_A := \sqrt{(\tilde{\mathbf{p}}-e\mathbf{A}_0(\tilde{\mathbf{x}}))^2 \,-e\boldsymbol{\sigma}\mathbf{B}_{A_0}(\tilde{\mathbf{x}})\,+\tilde{m}^2}, \quad \tilde{m} :=\; m/\mu_0^\delta.$$

If we write $h^H=\mu_0^\delta\,\tilde{h}^H$ with $\tilde{h}^H=\tilde{E}_A-\frac{\gamma}{\tilde{x}}$ and introduce the scaled ground-state energy $\tilde{E}_g=\mu_0^{-\delta}E_g$, the eigenvalue equation for h^H turns into

$$h^H\,\psi_g \;=\; E_g\,\psi_g \quad\Longleftrightarrow\quad \tilde{h}^H\,\psi_g \;=\; \tilde{E}_g\,\psi_g. \tag{4.3}$$

In the equation on the rhs of (4.3) the magnetic field strength enters only into the mass parameter $\tilde{m}$.

5. Existence of the ground state: Proof of Theorem 2(i)

Let us assume that $\mathbf{B}_A$ and V are fixed fields, determined by τ, B and Z, respectively, and let γ fulfil the condition of Theorem 2. We consider the operator $h_-^H = h^H - m$, define the ground-state energy without rest energy, $E_{g-} := E_g - m$, and show that for subcritical τ there exists some trial function $\psi_t \in L_2(\mathbb{R}^3) \otimes \mathbb{C}^2$ such that

$$(\psi_t, h_-^H\, \psi_t) \;<\; 0. \tag{5.1}$$

Since one has $E_{g-} = (\psi_g, h_-^H\, \psi_g) \leq (\psi_t, h_-^H\, \psi_t)$, this guarantees the existence of a discrete ground state. For further use we also define the scaled quantities $\tilde{h}_-^H := \tilde{h}^H - \tilde{m}$ and $\tilde{E}_{g-} := \tilde{E}_g - \tilde{m}$.

5.1. **Variational determination of an upper bound.**

We start by estimating the kinetic energy by an operator which allows for a separation of variables and thus simplifies the variational calculation.

Lemma 1. *If $\mathbf{A} \in L_{2,loc}(\mathbb{R}^3)$ is independent of x_3 and if $A_3 = 0$, we have for $\psi \in \mathcal{D}(E_A^{\frac{1}{2}})$,*

$$(\psi, E_A\, \psi) \;\leq\; \sqrt{(\psi, E_{xy}^2\, \psi)} \;+\; (\psi, \sqrt{p_3^2 + m^2}\, \psi), \tag{5.2}$$

$$E_{xy}^2 := \sum_{k=1}^{2} (p_k - eA_k)^2 \;-\; e\boldsymbol{\sigma}\mathbf{B}_A.$$

Proof. For any nonnegative operator $\mathcal{O} \geq 0$ and $\|\psi\| = 1$ we have

$$(\psi, \mathcal{O}\, \psi) \;\leq\; \|\psi\|\, \|\mathcal{O}\, \psi\| \;=\; \sqrt{(\psi, \mathcal{O}^2\, \psi)}. \tag{5.3}$$

Furthermore it is easily verified under the conditions of Lemma 1 that E_A^2 can be decomposed into the two nonnegative operators

$$E_A^2 \;=\; E_{xy}^2 \;+\; (p_3^2 + m^2). \tag{5.4}$$

Since E_{xy} and p_3 act on different coordinates they commute such that

$$(\psi, E_A\, \psi) \;\leq\; (\psi, E_{xy}\, \psi) \;+\; (\psi, \sqrt{p_3^2 + m^2}\, \psi) \tag{5.5}$$

and (5.2) follows from applying (5.3) to the first term of (5.5). □

For the trial function ψ_t we make the ansatz, guided by [18],

$$\psi_t(\mathbf{x}) \;=\; \psi_\perp(\varrho)\, \varphi_z(x_3) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varrho \;=\; \sqrt{x_1^2 + x_2^2}, \tag{5.6}$$

$$\psi_\perp(\varrho) \;=\; \tilde{N}\, e^{-s(eB)^d \varrho^2}, \qquad \tilde{N} \;=\; \sqrt{\frac{2s}{\pi}}\, (eB)^{\frac{d}{2}}$$

and

$$\varphi_z(x_3) \;=\; N_1\, e^{-Z'\sqrt{a_0^2 + x_3^2}}, \qquad N_1 \;=\; \frac{1}{\sqrt{2a_0 K_1(2a_0 Z')}}. \tag{5.7}$$

Both $\psi_\perp$ and φ_z are normalized to unity, and $s > 0$ and $Z' > 0$ are variational parameters. K_1 is a modified Bessel function. The power d of B in $\psi_\perp$ has to be chosen in a way that scaling is preserved, i.e. $B^d \varrho^2 = B_0^d \tilde{\varrho}^2$, resulting in $d = 2\delta = \frac{2}{2+\tau}$. Instead of taking φ_z to be the hydrogenic function $(e^{-Z'\sqrt{\varrho^2+x_3^2}})$ used in [18] we have replaced the ϱ-dependence by the constant

$$a_0 \;=\; \frac{1}{\sqrt{2s}\,(eB)^{\frac{1}{2+\tau}}} \tag{5.8}$$

in (5.7). This choice is only meaningful if the magnetic field strength dominates the strength of the Coulomb potential, i.e. for $B \gg Z^2 B_0$ [15, 10]. In that case the motion of the electron perpendicular to $\mathbf{B}_A$ is restricted to $\varrho \sim \left(s(eB)^d\right)^{-1/2} = \sqrt{2}a_0$. Therefore, we have in our calculations considered field strengths $B \in (2Z^2 B_0, \infty)$.

Now we calculate the rhs of (5.2) with $\psi = \psi_t$. With the help of the integral [6, p.337,1064]

$$\int_0^\infty dx\; x^q\, e^{-ax^2} \;=\; \frac{\Gamma(\frac{q+1}{2})}{2\,a^{\frac{q+1}{2}}}, \qquad q > -1, \quad a > 0, \tag{5.9}$$

the first term is given by

$$(\psi_t, E_{xy}^2\;\psi_t) \;=\; (\psi_\perp \begin{pmatrix}1\\0\end{pmatrix}, E_{xy}^2\;\psi_\perp \begin{pmatrix}1\\0\end{pmatrix}) \tag{5.10}$$

$$= \;(eB)^{\frac{2}{2+\tau}} \left(2s \;+\; \frac{\Gamma(\frac{3}{2}+\tau)}{\sqrt{\pi}\,s^{1+\tau}2^{2+\tau}} \;-\; (1+\tau)\,\frac{\Gamma(\frac{\tau+1}{2})}{\sqrt{\pi}\,(2s)^{\tau/2}}\right).$$

For the expectation value of the momentum operator we can use our previous results [10]. For its numerical computation it is convenient to introduce a slightly different, s-dependent scaling parameter ν. We define

$$\nu := \;\sqrt{2s}\;(eB)^{\frac{1}{2+\tau}}, \tag{5.11}$$

which has the same B-dependence as μ_0^δ. Then we introduce the scaled quantities

$$\tilde{Z} \;=\; Z'/\nu, \quad \tilde{a}_0 \;=\; a_0\nu \;=\; 1, \quad \tilde{m}_s \;=\; m/\nu, \quad \kappa \;=\; k/\nu. \tag{5.12}$$

Working in Fourier space by using

$$e^{-Z'\sqrt{a_0^2+x_3^2}} \;=\; \int_{-\infty}^\infty dk\;\tilde{f}(k)\;e^{ikx_3},$$

$$\tilde{f}(k) \;=\; \frac{a_0\,Z'}{\pi\sqrt{Z'^2+k^2}}\;K_1(a_0\sqrt{Z'^2+k^2}\,) \tag{5.13}$$

we have [10]

$$(\psi_t, \sqrt{p_3^2+m^2}\;\psi_t) \;=\; N_1 \int_{-\infty}^\infty dk\;\tilde{f}(k)\;\sqrt{k^2+m^2}\int_{-\infty}^\infty dx_3\;\varphi_z(x_3)\;e^{ikx_3} \tag{5.14}$$

$$= \;\frac{2\sqrt{2s}\tilde{a}_0\tilde{Z}^2(eB)^{\frac{1}{2+\tau}}}{\pi K_1(2\tilde{a}_0\tilde{Z})}\int_0^\infty d\kappa\;\frac{1}{\tilde{Z}^2+\kappa^2}\;K_1^2(\tilde{a}_0\sqrt{\tilde{Z}^2+\kappa^2}\,)\;\sqrt{\kappa^2+\tilde{m}_s^2}.$$

Finally we turn to the potential part of the Herbst operator. With the help of [6, p.339],

$$2\int_0^\infty \varrho\, d\varrho\; e^{-\nu^2\varrho^2}\;\frac{1}{\sqrt{\varrho^2+x_3^2}} \;=\; \frac{1}{\nu}\;E(\nu\,|x_3|\,),$$

$$E(y) \;=\; \sqrt{\pi}\; e^{y^2}\;[1-\phi(y)] \tag{5.15}$$

where ϕ is the probability function,

$$\phi(y) \;=\; \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}\;dt, \tag{5.16}$$

we get the following expression,

$$(\psi_t, -\frac{\gamma}{x}\;\psi_t) \;=\; -2\pi\gamma\;(\tilde{N}N_1)^2\int_0^\infty dx_3\; e^{-2Z'\sqrt{a_0^2+x_3^2}}\;\frac{1}{\nu}\;E(\nu\,x_3)$$

$$=\; -\gamma\;\frac{\sqrt{2s}(eB)^{\frac{1}{2+\tau}}}{\tilde{a}_0 K_1(2\tilde{a}_0\tilde{Z})}\int_0^\infty dy\;E(y)\;e^{-2\tilde{Z}\sqrt{\tilde{a}_0^2+y^2}}, \tag{5.17}$$

where the substitution $y=\nu x_3$ was used. We remark that our previous choice of $\tilde{a}=\frac{1}{\sqrt{2}}$ in place of $\tilde{a_0}$ [10] does not introduce any significant changes.

With the scaling (5.12), $\tilde{m}_s \to 0$ as $B\to\infty$ since s is only weakly dependent on B and remains nonzero (and finite) as $B\to\infty$. In fact for $\tau>0$ fixed, $(\psi_t, E_{xy}^2\,\psi_t)\to+\infty$ as $s\to 0$ while the other contributions to $E^H[\psi_t]$ vanish. In the other limit, $s\to\infty$, the upper bound for the kinetic energy increases like $s^{\frac{1}{2}}$. By using the estimate $x^{-1}\le\varrho^{-1}$ in the potential energy one has for $s\to\infty$, $\sqrt{(\psi_t, E_{xy}^2\,\psi_t)} - (\psi_t, \frac{\gamma}{x}\,\psi_t) \;\ge\; \sqrt{2s}(eB)^{\frac{1}{2+\tau}}(1-\epsilon-\gamma\sqrt{\pi})$, ϵ arbitrarily small, which is positive for $\gamma<\pi^{-\frac{1}{2}}$. Thus $E^H[\psi_t]$ tends to infinity both for $s=0$ and $s\to\infty$, yielding the minimum at finite s (this follows by numerical computation also for $\gamma\ge\pi^{-\frac{1}{2}}$).

From Lemma 1 we obtain

$$(\psi_t, h^H\;\psi_t) \;\le\; E^H[\psi_t] := \sqrt{(\psi_t, E_{xy}^2\;\psi_t)} \;+\; (\psi_t, \sqrt{p_3^2+m^2}\;\psi_t) \;-\; (\psi_t, \frac{\gamma}{x}\;\psi_t)$$

$$=: (eB)^{\frac{1}{2+\tau}}\;\tilde{E}^H[\psi_t], \tag{5.18}$$

where $\tilde{E}^H[\psi_t]$ depends only implicitly on B via $\tilde{m}_s$. (Note that for obtaining $E^H[\psi_t]$ in atomic units, the rhs of (5.18) has to be multiplied by $1/e^2\approx 137.04$, and $\mu_0^{\frac{1}{2+\tau}}$ has to be substituted for $(eB)^{\frac{1}{2+\tau}}$ with B measured in units of B_0.)

The desired upper bound of the scaled ground-state energy of h^H is thus given by

$$\tilde{E}_g^H := \min_{\tilde{Z}>0,\,s>0}\;\tilde{E}^H[\psi_t], \tag{5.19}$$

and again we set $\tilde{E}_-^H[\psi_t] = \tilde{E}^H[\psi_t]-\tilde{m}$, $\tilde{E}_{g-}^H = \tilde{E}_g^H - \tilde{m}$.

5.2. **Dependence on the field parameters.**

We start by proving that the Herbst operator has a discrete ground state for $\tau = 0$ when $0.1 \lesssim \gamma < 2/\pi$.

Lemma 2. *Let $\mathbf{B}_A = (0, 0, B)$ be constant. Then for a trial function ψ_t of the type (5.6) we have for $m = 0$,*

$$(\psi_t,\, h^H\, \psi_t) \;<\; (\psi_t,\, h^{BR}\, \psi_t), \tag{5.20}$$

where h^{BR} is the Brown-Ravenhall operator.

Proof. In the massless case the potential part of the Brown-Ravenhall operator h^{BR} from (1.2) reduces to

$$V^{BR} \;=\; -\frac{\gamma}{2}\,\left(\frac{1}{x} \;+\; \frac{\boldsymbol{\sigma}\mathbf{p}_A}{|\boldsymbol{\sigma}\mathbf{p}_A|}\,\frac{1}{x}\,\frac{\boldsymbol{\sigma}\mathbf{p}_A}{|\boldsymbol{\sigma}\mathbf{p}_A|}\right). \tag{5.21}$$

Representing φ_z in Fourier space and using the result of [10] for the expectation value of the nonlocal part of V^{BR} with a trial function of the type (5.6), we obtain

$$(\psi_t, V^{BR}\,\psi_t) \;=\; -\frac{\gamma}{2}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'\;\tilde{f}(k)\;\hat{V}_0(k-k')\left(1 \,+\, \frac{kk'}{|kk'|}\right)\,\tilde{f}(k') \tag{5.22}$$

with $\tilde{f}$ from (5.13) and

$$\hat{V}_0(q) \;=\; 4s\,N_1^2\;(eB)\int_{-\infty}^{\infty} dx_3\; e^{-iqx_3}\int_0^{\infty} \varrho\, d\varrho\; e^{-2s(eB)\,\varrho^2}\;\frac{1}{\sqrt{\varrho^2 + x_3^2}} \tag{5.23}$$

which is a positive symmetric function [11].

Using the same ψ_t, we have for the Coulomb potential instead

$$\begin{aligned}
(\psi_t, V\,\psi_t) \;&=\; -\gamma\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dk'\;\tilde{f}(k)\;\hat{V}_0(k-k')\;\tilde{f}(k') \\
&=\; -2\gamma\int_0^{\infty} dk\int_0^{\infty} dk'\;\tilde{f}(k)\;\tilde{f}(k')\;\left[\hat{V}_0(k-k') + \hat{V}_0(k+k')\right] \qquad (5.24)\\
&=\; (\psi_t, V^{BR}\,\psi_t) \;-\; 2\gamma\int_0^{\infty} dk\int_0^{\infty} dk'\;\tilde{f}(k)\;\tilde{f}(k')\;\hat{V}_0(k+k').
\end{aligned}$$

Since the integrand of the last term is nonnegative, this proves $(\psi_t, h^H\,\psi_t) \;<\; (\psi_t, h^{BR}\,\psi_t)$ in the massless case, irrespective of γ. □

Now we prove a monotonicity result for the Herbst operator.

Lemma 3. *Let $\tau \geq 0$ be fixed. Assume that to a given $Z = Z_0$, $\tilde{m} = \tilde{m}_0 \geq 0$, there exists a minimizing trial function $\psi_t(\tilde{Z}_0, s_0)$ of type (5.6) such that $\tilde{E}_-^H(Z_0, \tilde{m}_0)[\psi_t] < 0$. Then $\tilde{E}_-^H(Z_0, \tilde{m})[\psi_t] < 0$ for all $\tilde{m} \geq \tilde{m}_0$ and $\tilde{E}_-^H(Z, \tilde{m}_0)[\psi_t] < 0$ for all $Z \geq Z_0$, guaranteeing the existence of a discrete ground state for $Z \geq Z_0$ and $\tilde{m} \geq \tilde{m}_0$.*

Proof. Since by (3.15) $E_A - m$ is monotonically decreasing with m while the potential is independent of m, we have (keeping Z_0 and τ as well as the trial function fixed)

$$(\psi_t, \tilde{h}_-^H(Z_0, \tilde{m})\ \psi_t)\ \leq\ \tilde{E}_-^H(Z_0, \tilde{m})[\psi_t]\ <\ \tilde{E}_-^H(Z_0, \tilde{m}_0)[\psi_t]\ <\ 0 \qquad (5.25)$$

for all $\tilde{m} > \tilde{m}_0$. In a similar way (keeping $\tilde{m} \geq 0$ and τ fixed) one can profit from the fact that the potential decreases linearly with Z. Therefore

$$(\psi_t, \tilde{h}_-^H(Z, \tilde{m})\ \psi_t)\ \leq\ \tilde{E}_-^H(Z, \tilde{m})[\psi_t]\ <\ \tilde{E}_-^H(Z_0, \tilde{m})[\psi_t]\ <\ 0 \qquad (5.26)$$

for all $Z > Z_0$. These inequalities hold also for the minimum $\tilde{E}_{g-}^H$. So if a ground state exists for a set of parameters $\tau,\ \tilde{m}_0$ and Z_0, it also exists for $\tilde{m} > \tilde{m}_0$ and $Z > Z_0$ if τ is the same. $\square$

Since for $\tau = 0,\ \tilde{m} = 0$, it was shown [10, 11] that when $\gamma \in (0.1, \frac{2}{\pi})$ there exists a ψ_t of the type (5.6) such that $(\psi_t, h^{BR}\, \psi_t) < 0$, Lemma 2 guarantees the existence of a discrete ground state of h^H for $\tilde{m} = 0$ and the above range of γ. From Lemma 3 the ground state exists also for $\tilde{m} > 0$ (when $\tau = 0$).

We note that when $\tau = 0$ the expectation value (5.10) of E_{xy}^2 vanishes for $s = \frac{1}{4}$. In fact, the corresponding function $\psi_\perp$ turns into an exact eigenfunction to E_{xy}^2, with zero as eigenvalue, a well-known result for constant magnetic fields [20, p.196].

Table 1 gives the numerical upper bound of the ground-state energy at $\tau = 0$ and $\tilde{m} = 0$ for some nuclear charges Z. The corresponding variational parameters are $s = \frac{1}{4}$ and $\tilde{Z}$ as given in the Table. The comparison with the variational energies for the Brown-Ravenhall operator shows that the Herbst energies are indeed much lower.

Z	$\tilde{Z}$	$E_g^H/\sqrt{\mu_0}$	$E_g^{BR}/\sqrt{\mu_0}$
20	0.034	-0.907	-0.072
40	0.137	-6.032	-1.425
60	0.261	-14.345	-4.777
80	0.394	-24.716	-9.868

Table 1. Scaled variational ground-state energy $\tilde{E}_g^H(Z,0) = E_g^H/\sqrt{\mu_0}$ for a constant magnetic field $B = \mu_0 B_0$ (where $B_0 = 1\ [m^2e^3c/\hbar^3]\ = 2.35 \times 10^9$ G and $\mu_0 \to \infty$) and nuclear charge Z ranging from 20 to 80 together with the parameter $\tilde{Z}$. The last column gives the corresponding results for the Brown-Ravenhall operator [10] (in atomic units).

Table 2 compares the variational energies for finite constant magnetic fields with the Brown-Ravenhall results as well as with an available numerical result for the Dirac operator obtained by using elaborate trial functions [4]. In this context we recall that our trial function (5.6) does not provide an adequate bound for small magnetic fields ($\mu_0/Z^2 \lesssim 2$) because the separation of coordinates in $\psi_\perp(\varrho)\cdot\varphi_z(x_3)$ is then no longer a good approximation. For $Z = 20$ and $\mu_0/Z^2 = 2$ relativistic effects are small, and the Herbst operator and the Brown-Ravenhall operator give nearly the same results.

	$Z = 20$			$Z = 80$	
λ	$E^H_{g-}/\sqrt{\mu_0}$	$E^{BR}_{g-}/\sqrt{\mu_0}$	$E^{ex}_{g-}/\sqrt{\mu_0}$	$E^H_{g-}/\sqrt{\mu_0}$	$E^{BR}_{g-}/\sqrt{\mu_0}$
2	-12.74	-12.72	-14.48	-51.98	-50.86
10^2	-7.67	-7.49		-35.85	-30.88
10^4	-3.42	-2.93		-26.72	-15.62
10^6	-1.52	-0.89		-24.95	-10.84

Table 2. Ground-state energy (without rest energy) for electrons in a central Coulomb field ($Z = 20$ and 80) and in a constant magnetic field $B = \mu_0 B_0$ ($B_0 = 2.35 \times 10^9$ G). $\lambda := \mu_0/Z^2$ gives the ratio of magnetic and electric force exerted on the electron. Shown are the variational results for h^H and h^{BR} as well as an accurate numerical result [4] (in atomic units).

When $\tau \neq 0$, the τ-dependence of $\tilde{E}^H[\psi_t]$ is solely contained in the kinetic term (5.10). Although this term does not exhibit any monotonicity for $0 < \tau < 1$ (if s is fixed), we have shown numerically in the case $\tilde{m} = 0$ that $\tilde{E}^H_g$ increases strictly monotonically with τ. When τ is above 1 (and $s < \frac{1}{2}$ which is true for all $Z \leq 80$ in the minimum of $\tilde{E}^H[\psi_t]$) the kinetic energy and thus $\tilde{E}^H[\psi_t]$ increases strongly with τ. As a result the variational principle guarantees the existence of a negative-energy ground state only for sufficiently small τ (i.e. for a sufficiently weak increase of $\mathbf{B}_A$ at infinity). The 'critical' $\tau = \tau_c$ where $\tilde{E}^H_g$ reaches zero increases with nuclear charge Z (because, according to (5.26), a high Z lowers the energy). This is demonstrated in Fig.1. The existence of a ground state for infinitely strong magnetic fields and field parameters (Z, τ) to the right of the line shown in Fig.1 follows strictly from the monotonicity (5.26), since $0 = \tilde{E}^H_g(Z_0, 0) > \tilde{E}^H_g(Z, 0)$ for $\tau = \tau_c(Z_0)$ and $Z > Z_0$.

When the mass parameter is increased (or, equivalently, when the field strength B is reduced), τ_c increases too (according to (5.25)). Numerical results for the $\tilde{m}$-dependence of τ_c are shown in Fig.2. As a consequence of the monotonicity in $\tilde{m}$, $\tilde{E}_{g-}(Z, \tilde{m}_0) > \tilde{E}_{g-}(Z, \tilde{m})$ for $\tilde{m} > \tilde{m}_0$ and fixed τ, a bound state exists for all pairs $(\tau, \tilde{m})$ to the right of the curve in Fig.2. In the two-dimensional parameter space $(Z, \tilde{m})$ (respectively (Z, B)) τ_c thus spans a surface below which a ground state of h^H is guaranteed.

Fig.3 shows the B-dependence of $E_{g-}^H = \tilde{E}_{g-}^H \cdot (eB)^{1/(2+\tau)}$ for $Z = 80$ at several values of τ. The higher τ, the lower is the critical B where E_{g-}^H reaches zero. For e.g. $\tau = 4$, the existence of a ground state of h^H is only guaranteed for $B < 10^4\, Z^2 B_0 = 1.5 \times 10^{17}$ G.

6. High-field limit of the ground-state energy: proof of Theorem 2(ii)

From Theorem 2(i) we know that for fixed $\gamma \in (0.1, \frac{2}{\pi})$ there exists $\tau_c > 0$ such that, when $\tilde{m} = 0$, $\tilde{h}^H$ has a discrete ground state with energy $\tilde{E}_g(0) < 0$ for $\tau < \tau_c$. Moreover, there exists a sequence $(\tilde{m}_n)_{n \in \mathbb{N}}$ converging to zero which generates a sequence $(\tilde{E}_{g-}(\tilde{m}_n))_{n\in\mathbb{N}}$ of ground-state energies for the same γ and τ. In order to obtain the behaviour of the ground-state energy of h^H we have to prove that $(\tilde{E}_{g-}(\tilde{m}_n))_{n\in\mathbb{N}}$ converges to $\tilde{E}_g(0)$, or equivalently, that there exists an $m_0 > 0$ such that

$$|\tilde{E}_{g-}(\tilde{m}) - \tilde{E}_g(0)| \; < \; \epsilon \qquad \text{for all } \tilde{m} \; < \; m_0. \tag{6.1}$$

In fact, since $h_-^H(m)$ is monotonically decreasing with m according to (3.15), we have $h_-^H(m) \leq h^H(0)$ and hence $E_{g-}(m) \leq E_g(0)$. On the other hand it follows from the definition (1.3) of E_A that $h_-^H(m) + m \geq h^H(0)$ such that $E_{g-}(m) + m \geq E_g(0)$. Combining the two inequalities, we have

$$-m \; \leq \; E_{g-}(m) - E_g(0) \; \leq \; 0, \tag{6.2}$$

from which (6.1) is an immediate consequence. It then follows from the scaling that

$$\tilde{E}_g(0) \; = \; \lim_{\tilde{m}\to 0} \tilde{E}_{g-}(\tilde{m}) \; = \; \lim_{\mu_0 \to \infty} \; \mu_0^{-\delta} E_g. \tag{6.3}$$

With $\mu_0 = B/B_0$ this leads to the asymptotic behaviour,

$$E_g \; \sim \; \mu_0^\delta \; \tilde{E}_g(0) \; = \; -c\, B^\delta, \tag{6.4}$$

where $c = |\tilde{E}_g(0)|\, B_0^{-\delta}$ and $\delta = \frac{1}{2+\tau}$. We have $c > 0$ since the upper bound for $\tilde{E}_g(0)$ is negative when $\tau < \tau_c$.

Note that for $\tau = 0$, $E_g \sim -c\sqrt{B}$, which coincides with the behaviour of the ground-state energy of the Brown-Ravenhall operator [11].

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Figure Captions

Fig.1
Critical field growth τ_c providing an upper bound zero for the ground-state energy at $\tilde{m} = 0$ as a function of central charge Z.

Fig.2
Critical field growth τ_c for $Z = 20$ $(- - -)$ and 80 (——-) as a function of the mass parameter $\tilde{m}$. The end point to the right of each curve corresponds to $\lambda = 2$ (the minimum field strength considered).

Fig.3
Ground-state energy E_{g-}^H (in atomic units; without rest energy) as a function of μ_0/Z^2 for $Z = 80$ and $\tau = 0.5$ $(- - -)$, 2 (——-) and 4 $(- \cdot - \cdot -)$. In order to allow for a common display the energies corresponding to $\tau = 0.5$ are divided by 10 in the plot.

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